2-FAST (2-point function from Fast and Accurate Spherical Bessel Transformation)

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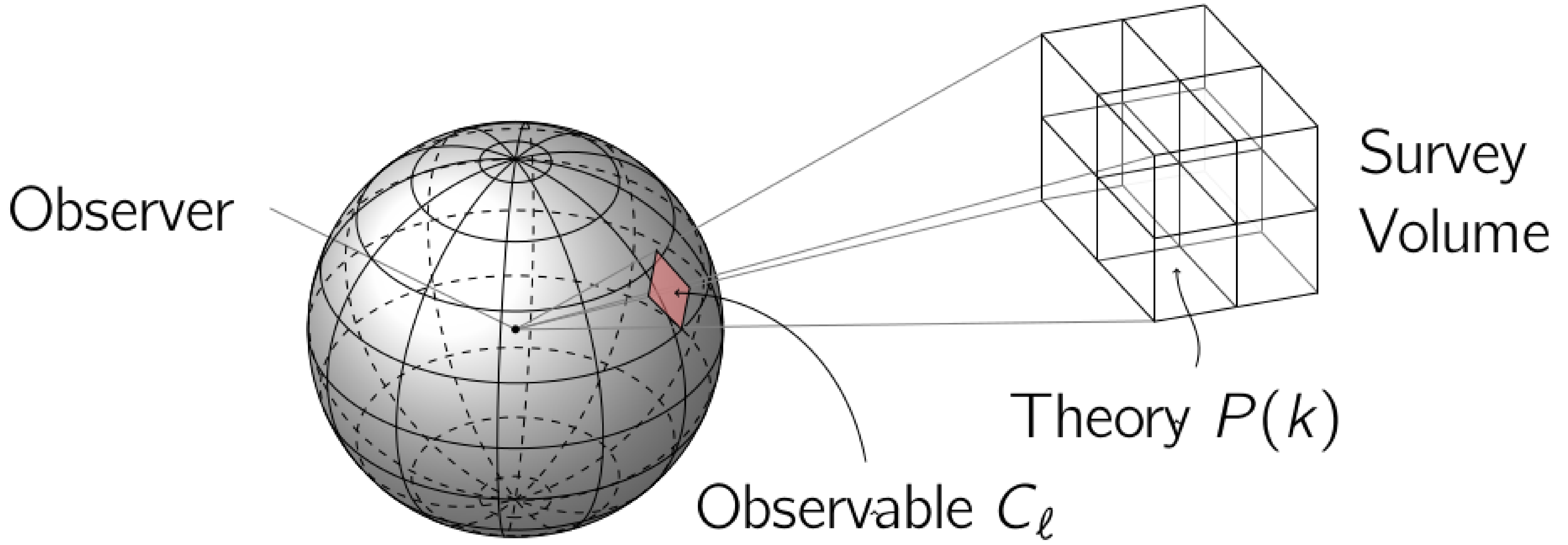
Two Integrals

Projection onto real space:

$$\xi_\ell^\nu(r) \equiv \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2} \, P(k) \, \frac{j_\ell(kr)}{(kr)^\nu}$$

Projection onto spherical harmonic space:

$$w_{\ell\ell'}(\chi, \chi') = \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, k^2 \, P(k) \, j_\ell(k\chi) \, j_{\ell'}(k\chi')$$



Perturbation Theory

$$\xi_\ell^\nu(r) \equiv \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2} \, P(k) \, \frac{j_\ell(kr)}{(kr)^\nu}$$

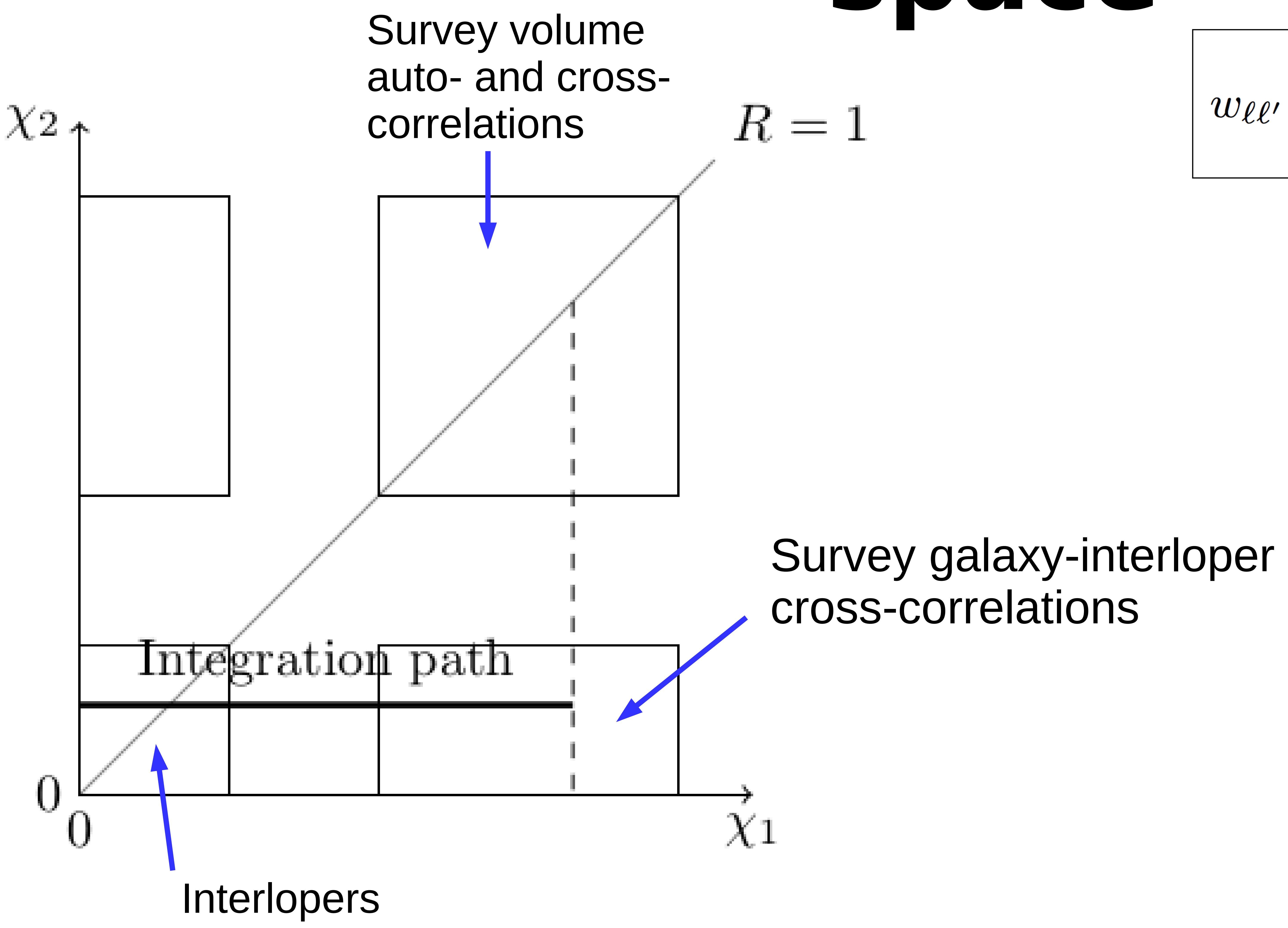
Projection onto real space:

$$\xi_{22}(r) = \frac{1219}{735}[\xi_0^0(r)]^2 + \frac{1}{3}\xi^0_{-2}(r)\xi_2^0(r) - \frac{124}{35}\xi^1_{-1}(r)\xi_1^1(r) + \frac{1342}{1029}[\xi_0^2(r)]^2 + \frac{2}{3}\xi^2_{-2}(r)\xi_2^2(r) - \frac{16}{35}\xi^3_{-1}(r)\xi_1^3(r) + \frac{64}{1715}[\xi_0^4]^2$$

$$\begin{aligned} P_{13}(k) = P_L(k)\Bigg[& \frac{67k^2}{189}\int dr\, r\, j_0(kr)\,\xi_0^0(r) - \frac{k^4}{3}\int drrj_0(kr)\xi^0_{-2}(r) + \frac{227k^3}{315}\int drrj_1(kr)\xi^1_{-1}(r) \\ & - \frac{37k}{45}\int drrj_1(kr)\xi_1^1(r) - \frac{2k^4}{3}\int drrj_2(kr)\xi^2_{-2}(r) - \frac{46k^2}{189}\int drrj_2(kr)\xi_0^2(r) \\ & + \frac{76k^3}{105}\int drrj_3(kr)\xi^3_{-1}(r) + \frac{4k}{15}\int drrj_3(kr)\xi_1^3(r)\Bigg] \end{aligned}$$

Schmittfull et al 2016, McEwen et al 2016, Fang et al 2016

Projection onto spherical harmonic space



$$w_{\ell\ell'}(\chi, \chi') = \frac{2}{\pi} \int_0^\infty \mathrm{d}k\, k^2\, P(k)\, j_\ell(k\chi)\, j_{\ell'}(k\chi')$$

Lensing magnification is the dominant cross-correlation between galaxy samples with large separation in redshift.

Linear redshift-space distortions

$$w_{\ell\ell'}(\chi,\chi') = \frac{2}{\pi}\int_0^\infty \mathrm{d}k\, k^2\, P(k)\, j_\ell(k\chi)\, j_{\ell'}(k\chi')$$

- Need w_ℓ with second derivative on the spherical Bessels
- Can be expressed as a linear combination of $w_{\ell\ell'}$ with $\ell' = \ell \pm 2$.

→ Lot's of $w_{\ell\ell'}$ are needed!

$$\begin{pmatrix} w_{\ell-2,\ell-2} & w_{\ell-2,\ell} & w_{\ell-2,\ell+2} \\ w_{\ell,\ell-2} & w_{\ell,\ell} & w_{\ell,\ell+2} \\ w_{\ell+2,\ell-2} & w_{\ell+2,\ell} & w_{\ell+2,\ell+2} \end{pmatrix}$$

We don't want to spend a week on a supercomputer to compute these.

A minute on a laptop is better!

Problems with the two integrals

$$\xi_\ell^\nu(r) \equiv \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2}\, P(k)\, \frac{j_\ell(kr)}{(kr)^\nu}$$

$$w_{\ell\ell'}(\chi,\chi') = \frac{2}{\pi}\int_0^\infty \mathrm{d}k\, k^2\, P(k)\, j_\ell(k\chi)\, j_{\ell'}(k\chi')$$

- Spherical Bessel functions are Oscillatory, and decay slowly (~1/x).
- Coefficients are exponentially complex for higher ℓ. (need ℓ~1000)
- Integration must be done from 0 to infinity, that is, over many many oscillations.

In[1]:=
```
Table[SphericalBesselJ[n, x] // FunctionExpand, {n, 0, 17}]
```

Out[1]=

$$\Big\{\frac{\sin x}{x},\; -\frac{\cos x}{x}+\frac{\sin x}{x^2},\; -\frac{3\cos x}{x^2}+\frac{(3-x^2)\sin x}{x^3},\; \frac{(-15+x^2)\cos x}{x^3}-\frac{3(-5+2x^2)\sin x}{x^4},\; \frac{5(-21+2x^2)\cos x}{x^4}+\frac{(105-45x^2+x^4)\sin x}{x^5},$$

$$\frac{(-945+105x^2-x^4)\cos x}{x^5}+\frac{15(63-28x^2+x^4)\sin x}{x^6},\; -\frac{21(495-60x^2+x^4)\cos x}{x^6}+\frac{(10395-4725x^2+210x^4-x^6)\sin x}{x^7},$$

$$\frac{(-135135+17325x^2-378x^4+x^6)\cos x}{x^7}-\frac{7(-19305+8910x^2-450x^4+4x^6)\sin x}{x^8},\; \frac{9(-225225+30030x^2-770x^4+4x^6)\cos x}{x^8}+\frac{(2027025-945945x^2+51975x^4-630x^6+x^8)\sin x}{x^9},$$

$$\frac{(-34459425+4729725x^2-135135x^4+990x^6-x^8)\cos x}{x^9}+\frac{45(765765-360360x^2+21021x^4-308x^6+x^8)\sin x}{x^{10}},$$

$$-\frac{55(11904165-1670760x^2+51597x^4-468x^6+x^8)\cos x}{x^{10}}+\frac{(654729075-310134825x^2+18918900x^4-315315x^6+1485x^8-x^{10})\sin x}{x^{11}},$$

$$\frac{(-13749310575+1964187225x^2-64324260x^4+675675x^6-2145x^8+x^{10})\cos x}{x^{11}}-\frac{33(-416645775+198402750x^2-12530700x^4+229320x^6-1365x^8+2x^{10})\sin x}{x^{12}},$$

$$\frac{39(-8108567775+1175154750x^2-40291020x^4+471240x^6-1925x^8+2x^{10})\cos x}{x^{12}}+$$

$$\frac{(316234143225-151242416325x^2+9820936125x^4-192972780x^6+1351350x^8-3003x^{10}+x^{12})\sin x}{x^{13}},$$

$$\frac{(-7905853580625+1159525191825x^2-41247931725x^4+523783260x^6-2552550x^8+4095x^{10}-x^{12})\cos x}{x^{13}}+$$

$$\frac{91(86877511875-41701205700x^2+2770007625x^4-57558600x^6+454410x^8-1320x^{10}+x^{12})\sin x}{x^{14}},$$

$$-\frac{105(2032933777875-301175374500x^2+11043097065x^4-149652360x^6+831402x^8-1768x^{10}+x^{12})\cos x}{x^{14}}+$$

$$\frac{(213458046676875-102776096548125x^2+6957151150950x^4-151242416325x^6+1309458150x^8-4594590x^{10}+5460x^{12}-x^{14})\sin x}{x^{15}},$$

$$\frac{(-6190283353629375+924984868933125x^2-34785755754750x^4+496939367925x^6-3055402350x^8+7936110x^{10}-7140x^{12}+x^{14})\cos x}{x^{15}}-$$

$$\frac{15(-412685556908625+199227510231750x^2-13703479539750x^4+309206717820x^6-2880807930x^8+11639628x^{10}-18564x^{12}+8x^{14})\sin x}{x^{16}},$$

$$\frac{17(-11288163762500625+1699293469623750x^2-65293049571750x^4+974390917500x^6-6495939450x^8+19606860x^{10}-23940x^{12}+8x^{14})\cos x}{x^{16}}+$$

$$\frac{(191898783962510625-92854250304440625x^2+6474894082531875x^4-150738274937250x^6+1490818103775x^8-6721885170x^{10}+13226850x^{12}-9180x^{14}+x^{16})\sin x}{x^{17}},$$

$$\frac{(-6332659870762850625+959493919812553125x^2-37554385678684875x^4+581419060472250x^6-4141161399375x^8+14054850810x^{10}-21366450x^{12}+11628x^{14}-x^{16})\cos x}{x^{17}}+$$

$$\frac{153(41389933795835625-20067846688890000x^2+1416077891353125x^4-33855655333500x^6+351863386875x^8-1732250520x^{10}+3993990x^{12}-3800x^{14}+x^{16})\sin x}{x^{18}}\Big\}$$

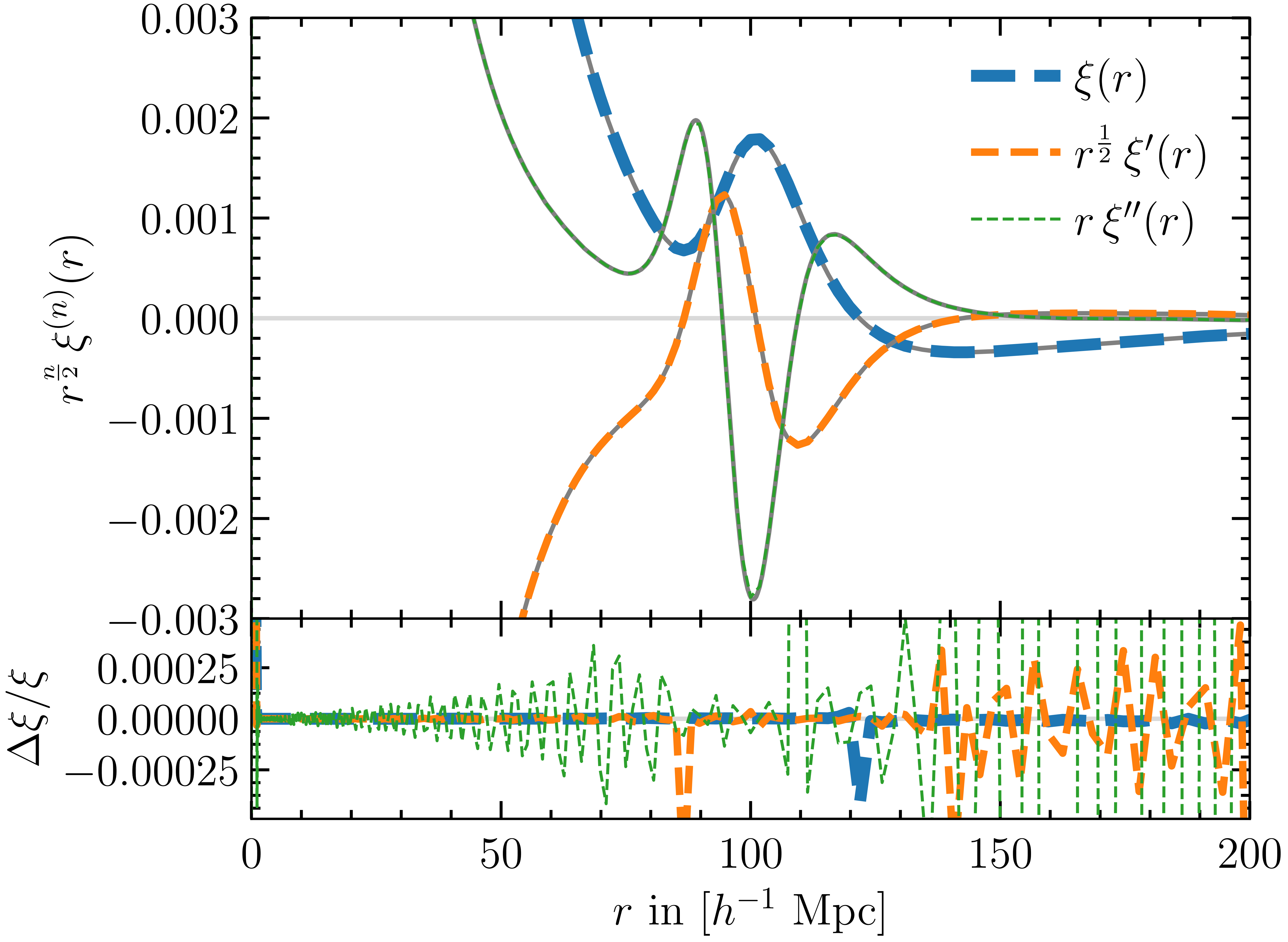
Need up to ℓ ≈ 1000

Key Idea: Projection is Convolution!

- Introduce logarithmic variables $\rightarrow$ Convolution integral!
- Convolution is Multiplication in Fourier-space
- Can calculate convolution kernel analytically
- Use FFT

Talman 1978

The first integral: ~1.3ms on laptop!



$$\xi'(r) = -\frac{1}{r}\xi_1^{-1}(r)$$

$$\xi''(r) = \frac{1}{r^2}\left[\xi_2^{-2}(r) - \xi_1^{-1}(r)\right]$$

The second integral

Projection onto spherical harmonic space:

$$w_{\ell\ell'}(\chi, \chi') \equiv \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, k^2 \, P(k) \, j_\ell(k\chi) \, j_{\ell'}(k\chi')$$

Spherical Bessel functions $j_\ell(kr)$ are highly oscillatory!

The second integral

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Spherical Bessel functions $j_\ell(kr)$ are highly oscillatory!

Defining **logarithmic** variables κ, ρ, and R s.t.

$$k = k_0 \, e^\kappa \qquad \chi = \chi_0 \, e^\rho \qquad \chi' = R\chi \, ,$$

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Eq. (1) becomes a **convolution** type integral:

$$w_{\ell\ell'}(\chi, R\chi) = \frac{2k_0^3}{\pi}\, e^{-q\rho} \int_{-\infty}^{\infty} \mathrm{d}\kappa \left[e^{(3-q)\kappa}\, P(k_0 e^\kappa)\right] \times \left[e^{q(\kappa+\rho)} j_\ell(k_0\chi_0 e^{\kappa+\rho})\, j_{\ell'}(k_0 R\chi_0 e^{\kappa+\rho})\right]$$

q is a biasing parameter.

The second integral

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q is a biasing parameter.
Convolution is Multiplication! Define $\phi^q(t)$ and $M^q_{\ell\ell'}(t,R)$ s.t.

$$e^{(3-q)\kappa}\, P(k_0 e^\kappa) = \int \mathrm{d}t\, e^{-i\kappa t}\, \phi^q(t)$$

$$e^{q\sigma}\, j_\ell(k_0\chi_0 e^\sigma)\, j_{\ell'}(k_0 R\chi_0 e^\sigma) = \int \frac{\mathrm{d}t}{2\pi}\, e^{i\sigma t}\, M^q_{\ell\ell'}(t,R)\,.$$

The second integral

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The 2-FAST core:

$$w_{\ell\ell'}(\chi, R\chi) = 4k_0^3\, e^{-q\rho}\int_{-\infty}^{\infty}\frac{dt}{2\pi}\, e^{i\rho t}\, \phi^q(t)\, M^q_{\ell\ell'}(t,R)\,.$$

$$\begin{aligned} M^q_{\ell\ell'}(t,R) &= \int d\sigma\, e^{(q-it)\sigma}\, j_\ell(\alpha e^\sigma)\, j_{\ell'}(\beta e^\sigma) \\ &= \alpha^{-1}\int ds \left(\frac{s}{\alpha}\right)^{q-1-it} j_\ell(s)\, j_{\ell'}(Rs) \\ &= \alpha^{it-q}\int ds\, s^{q-1-it}\, j_\ell(s)\, j_{\ell'}(Rs) \\ &= \alpha^{it-q}\, U_{\ell\ell'}(R, q-1-it) \end{aligned} \qquad (26)$$

where $s = \alpha e^\sigma$, or $\sigma = \ln(s/\alpha)$, and $U_{\ell\ell'}(R,n)$ is given in terms of the Gauss hypergeometric function ${}_2F_1$ as

$$\begin{aligned} &U_{\ell\ell'}(R,n) \\ &= 2^{n-2} R^{\ell'}\pi\, \frac{\Gamma[(1+\ell+\ell'+n)/2]}{\Gamma[(2+\ell-\ell'-n)/2]\,\Gamma[\frac{3}{2}+\ell']} \\ &\quad\times {}_2F_1\left(\frac{-\ell+\ell'+n}{2}, \frac{1+\ell+\ell'+n}{2}; \frac{3}{2}+\ell'; R^2\right). \end{aligned} \qquad (27)$$

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Talman 1978 had ℓ = ℓ'

ℓ ≠ ℓ' needed for redshift-space distortion

How to calculate $M^q_{\ell\ell'}(t,R)$ for large ℓ?

The second integral

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Gauss hypergeometric function

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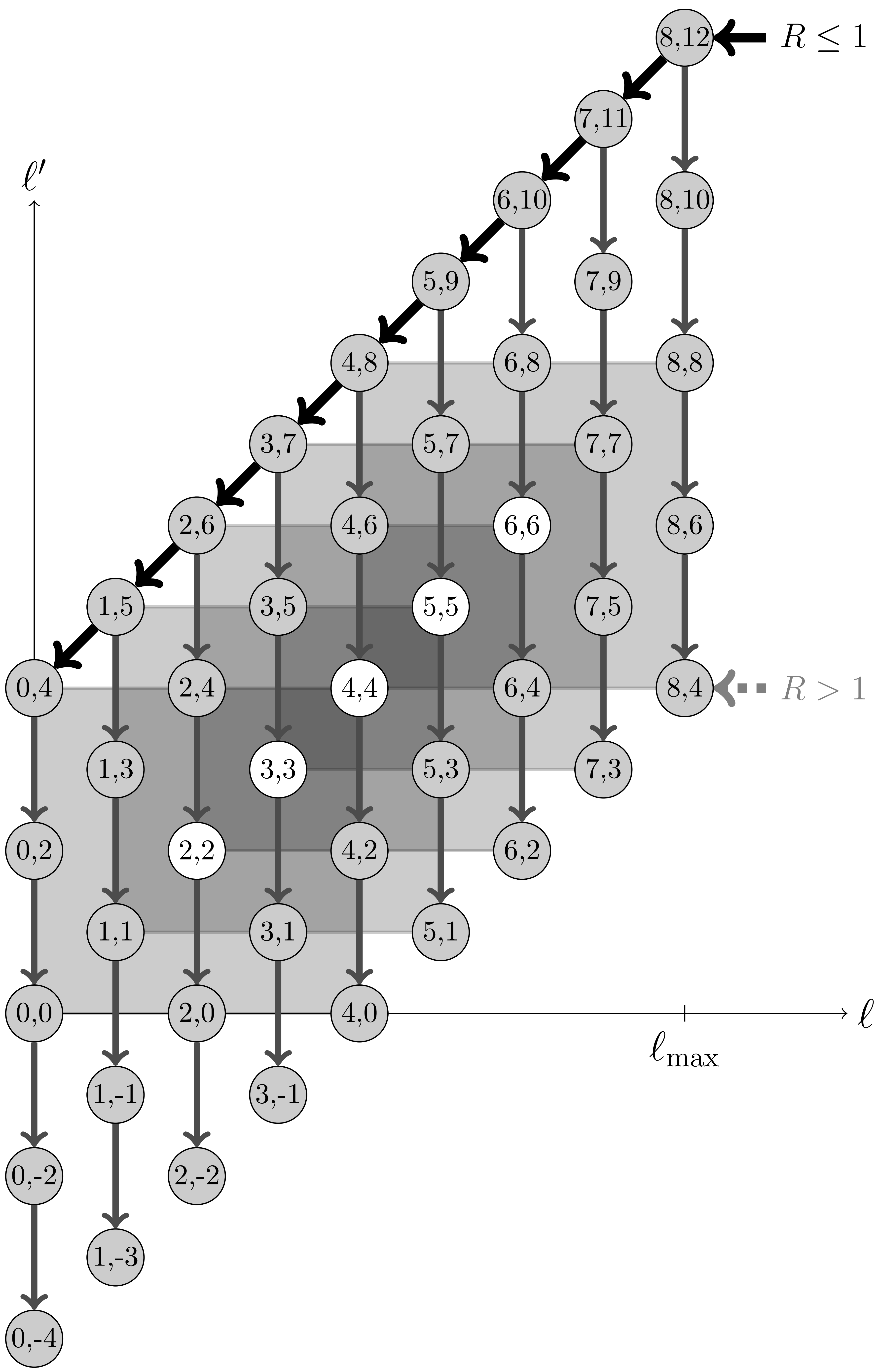
ℓ ≠ ℓ’ needed for redshift-space distortion

How to calculate $M^q_{\ell\ell'}(t,R)$ for large ℓ?

Walking in (ℓ,ℓ')-space



Recursion relations:

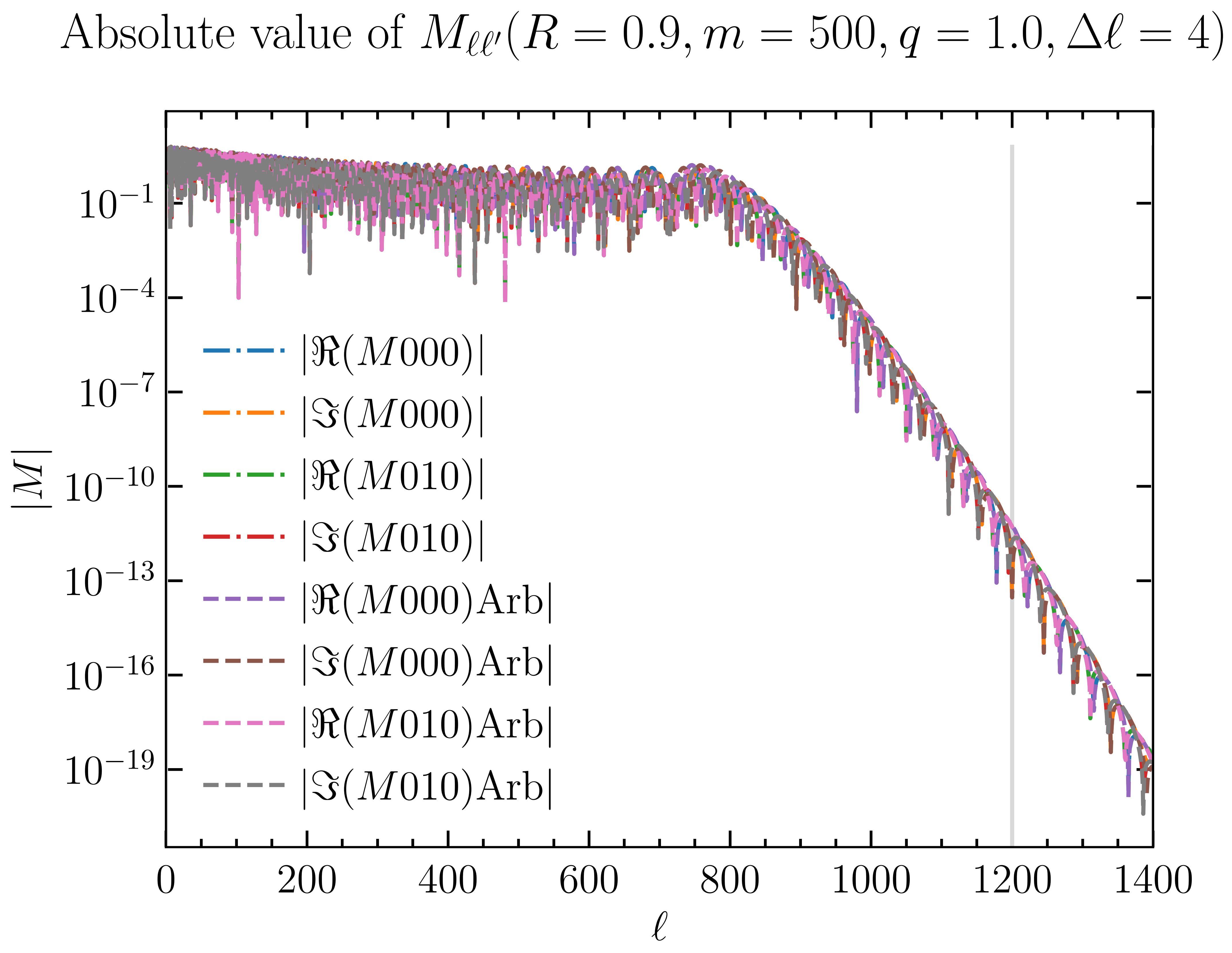
Unstable: $\ell \longrightarrow \ell + 1$

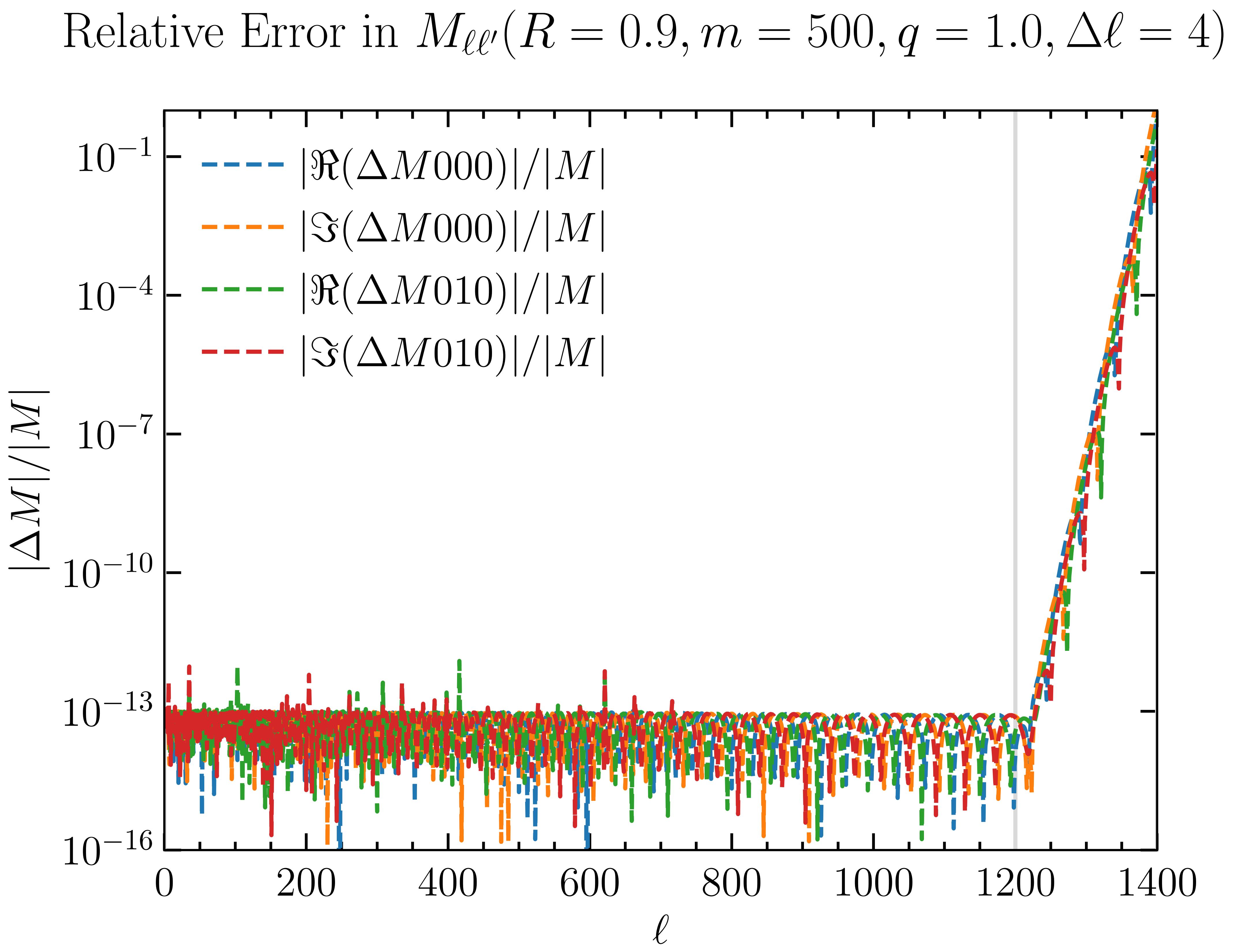
Stable: $\ell \longrightarrow \ell - 1$

Stable: $\Delta\ell \longrightarrow \Delta\ell - 2$ if $R < 1$

Stable: $\Delta\ell \longrightarrow \Delta\ell + 2$ if $R > 1$

Miller’s algorithm: Backwards Recursion





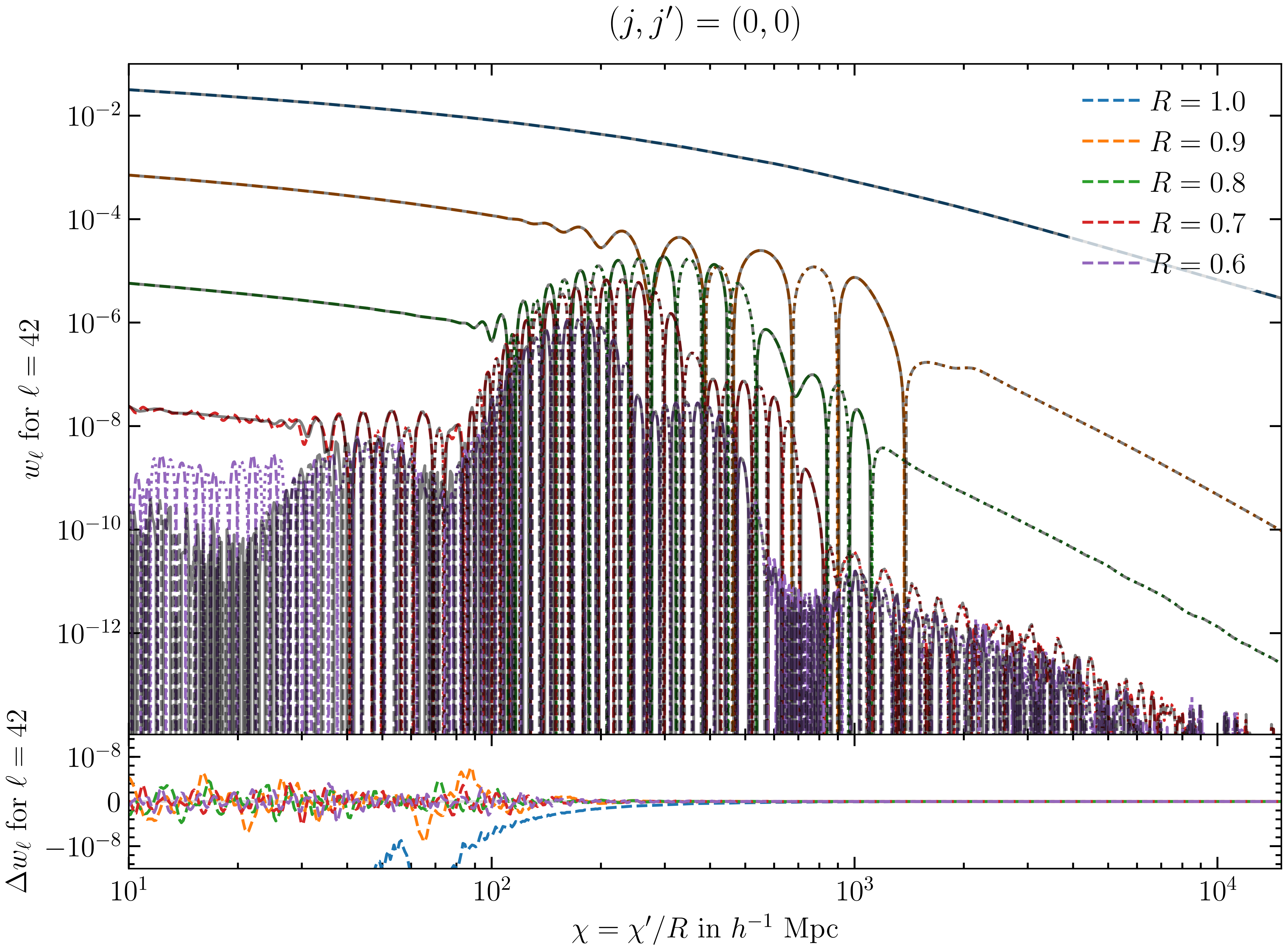
Recursion has two solutions.

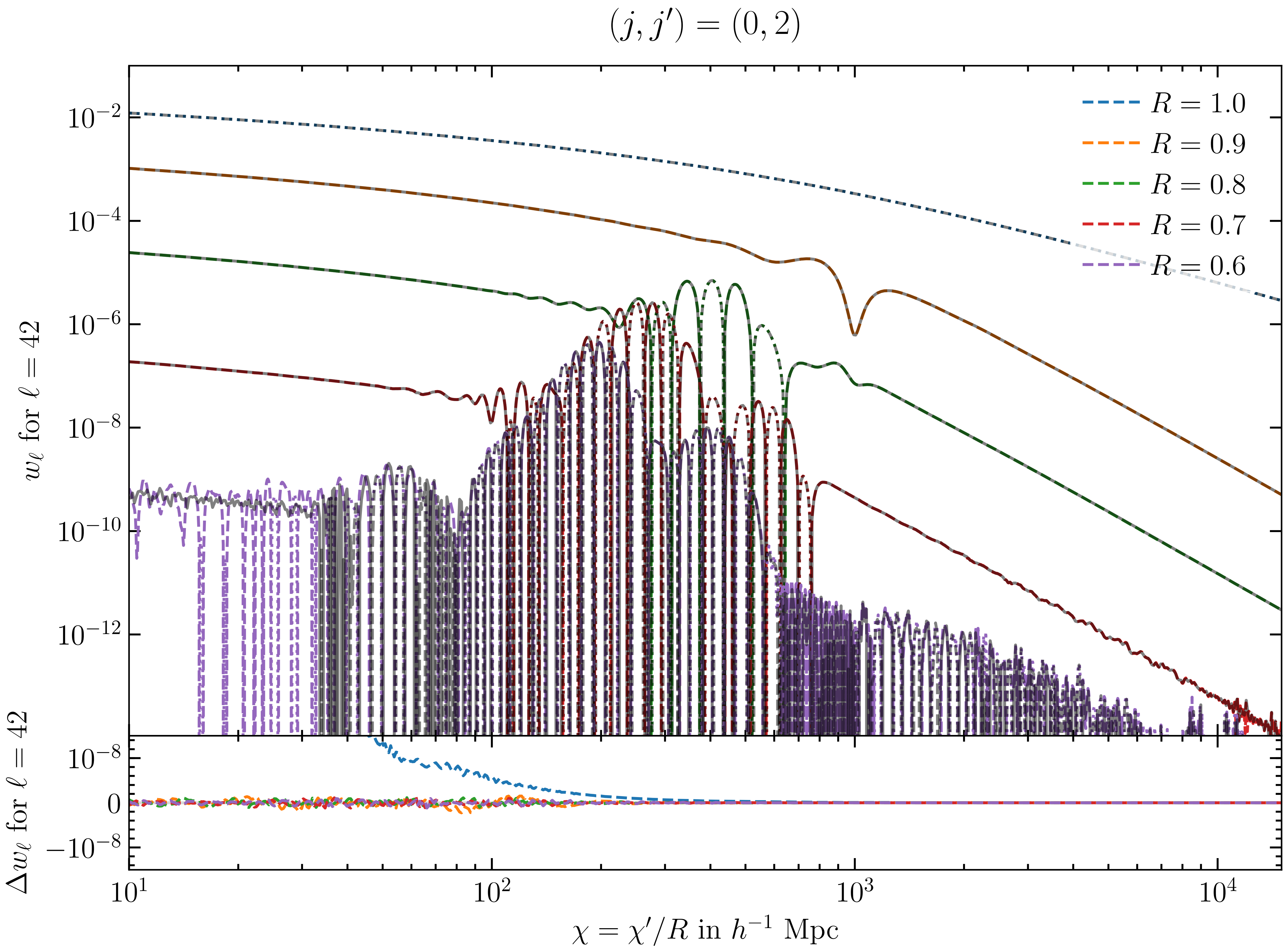
Forward direction: error solution grows faster than $M^q_{\ell\ell'}$ → unstable

Backward direction: $M^q_{\ell\ell'}$ grows faster than error solution → stable!

Match at $\ell = 0$.

Accurate





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Comparison is with Lucas 1995

And fast!

TABLE I. Performance results.

N[a]	N_χ[b]	N_R[c]	ℓ_{max}	${}_2F_{1,\ell_{max}}$	$M^q_{\ell\ell'}$	C_ℓ	Total[d]	IO[e]
1600	1	1	500	326 ms	215 ms	28 ms	569 ms	68 ms
1600	1	1	1200	393 ms	446 ms	60 ms	899 ms	142 ms
1600	1600	1	1200	404 ms	453 ms	69 ms	926 ms	163 ms[f]
3200	3200	5	1200	3.85 s	3.44 s	0.45 s	7.74 s	1.10 s

[a] Number of sample points on the power spectrum $P(k)$

[b] Number of redshifts, or number of χ

[c] Number of ratios $R = \chi'/\chi$

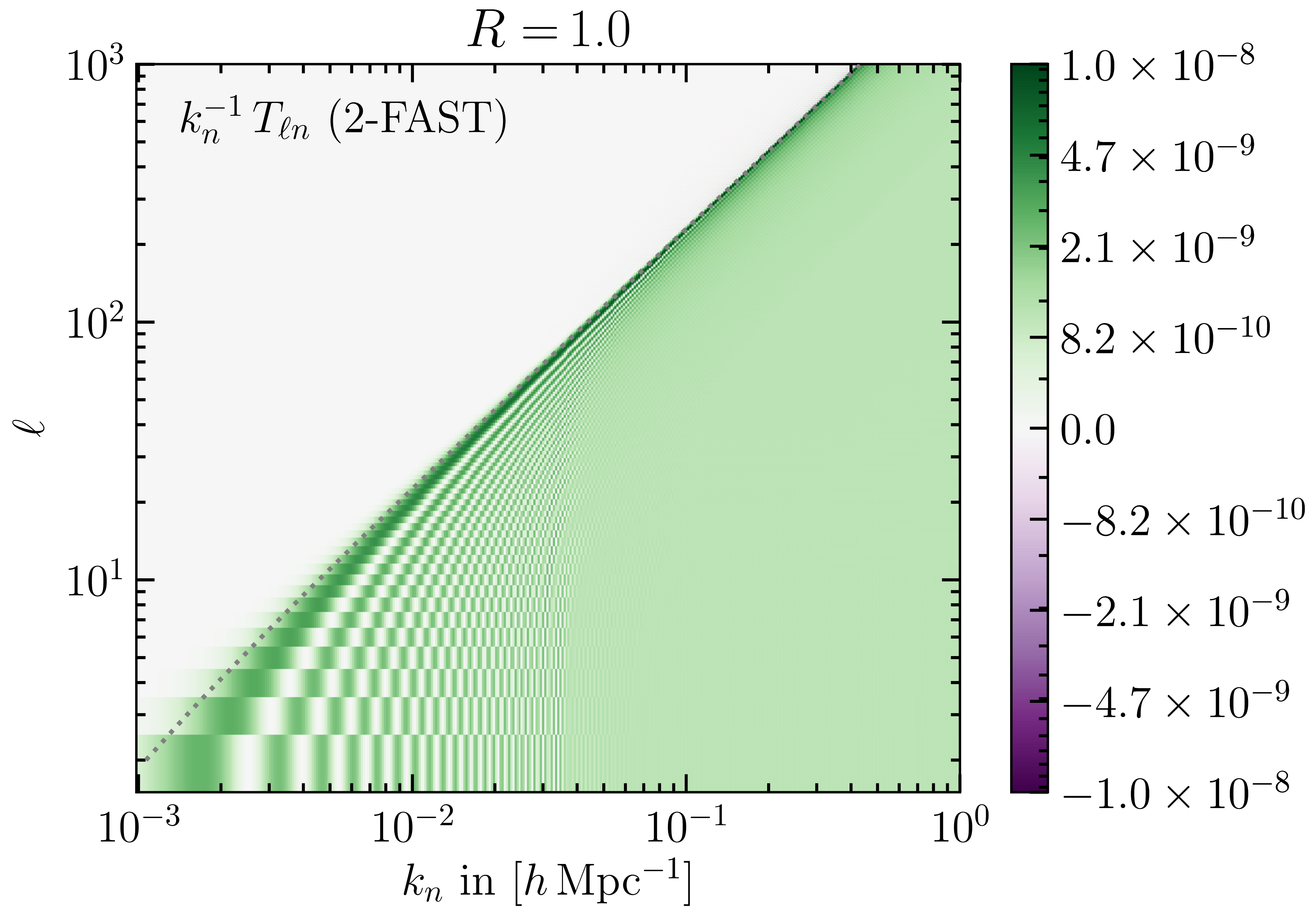
[d] Sum of the three preceding times

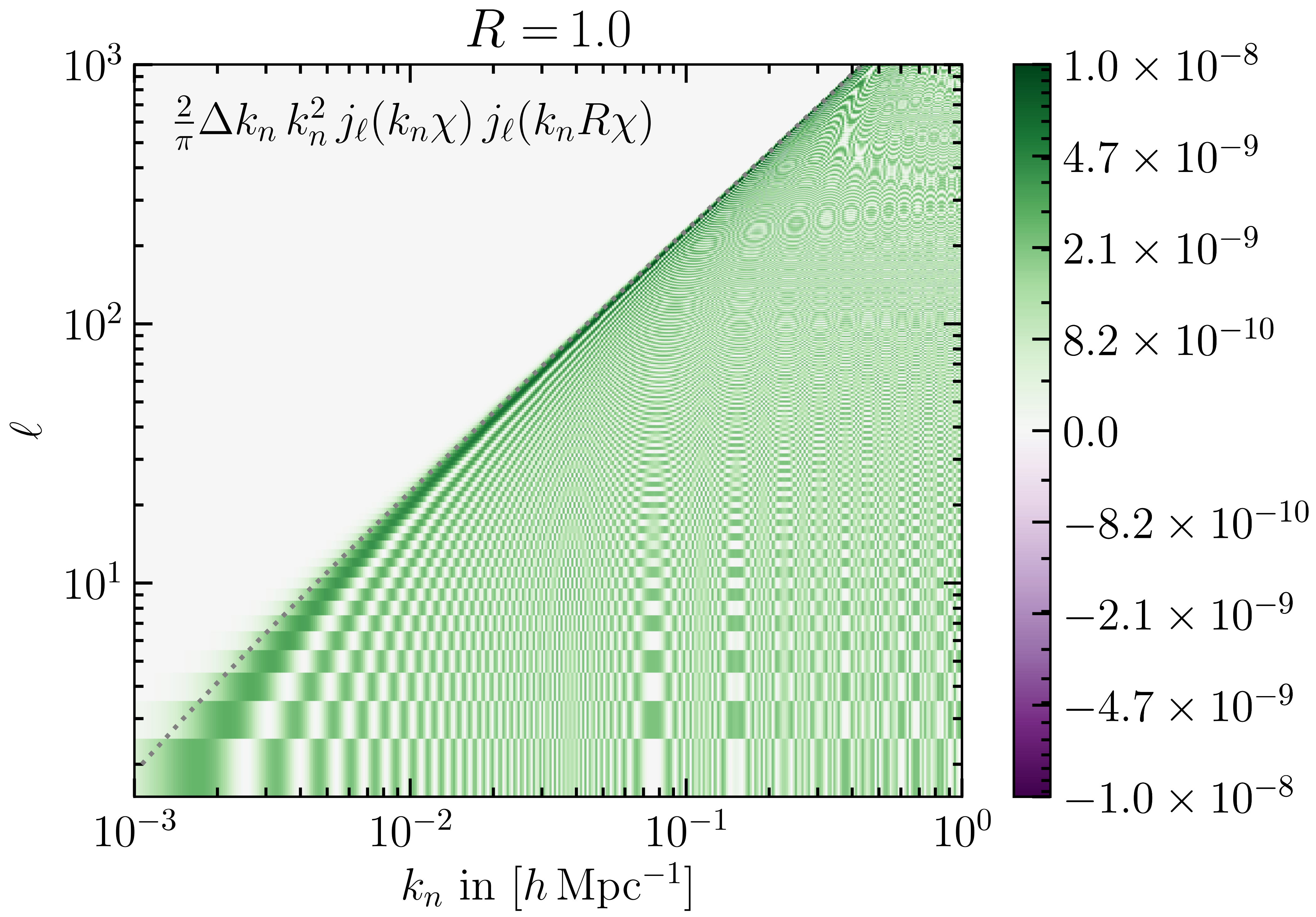
[e] Time spent reading and writing to disk

[f] Since we are only interested in compute times here, we did not save all 1600 values to disk in this case.

Why it works

Map k-space to ℓ-space



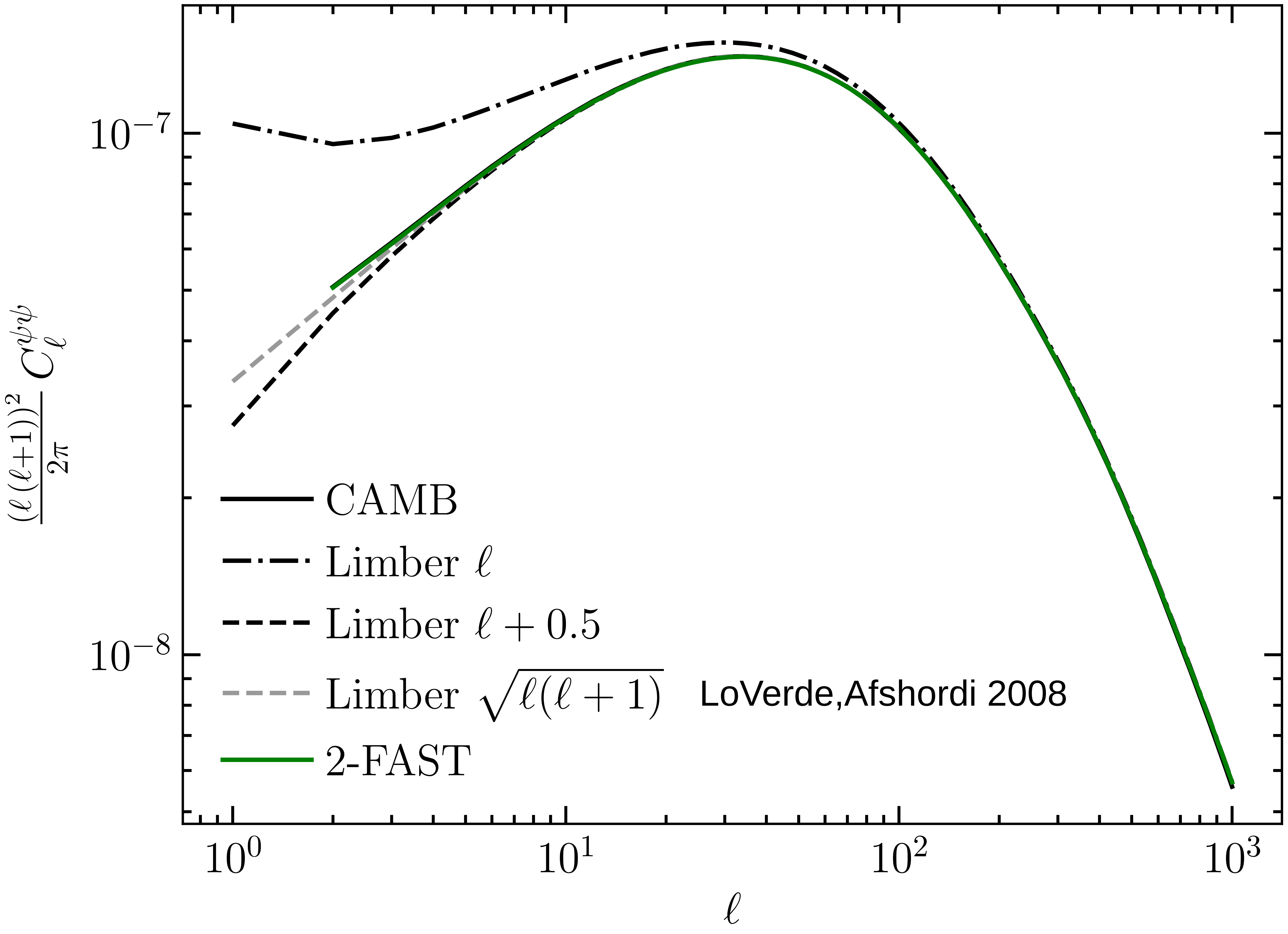


2-FAST

Traditional numerical integration:

$$w_{\ell\ell'}(\chi, R) = \sum_n \left[\frac{2}{\pi}\Delta k_n\, k_n^2\, j_\ell(k_n\chi)\, j_{\ell'}(k_n R\chi)\right] P(k_n)\,.$$

Example: CMB lensing



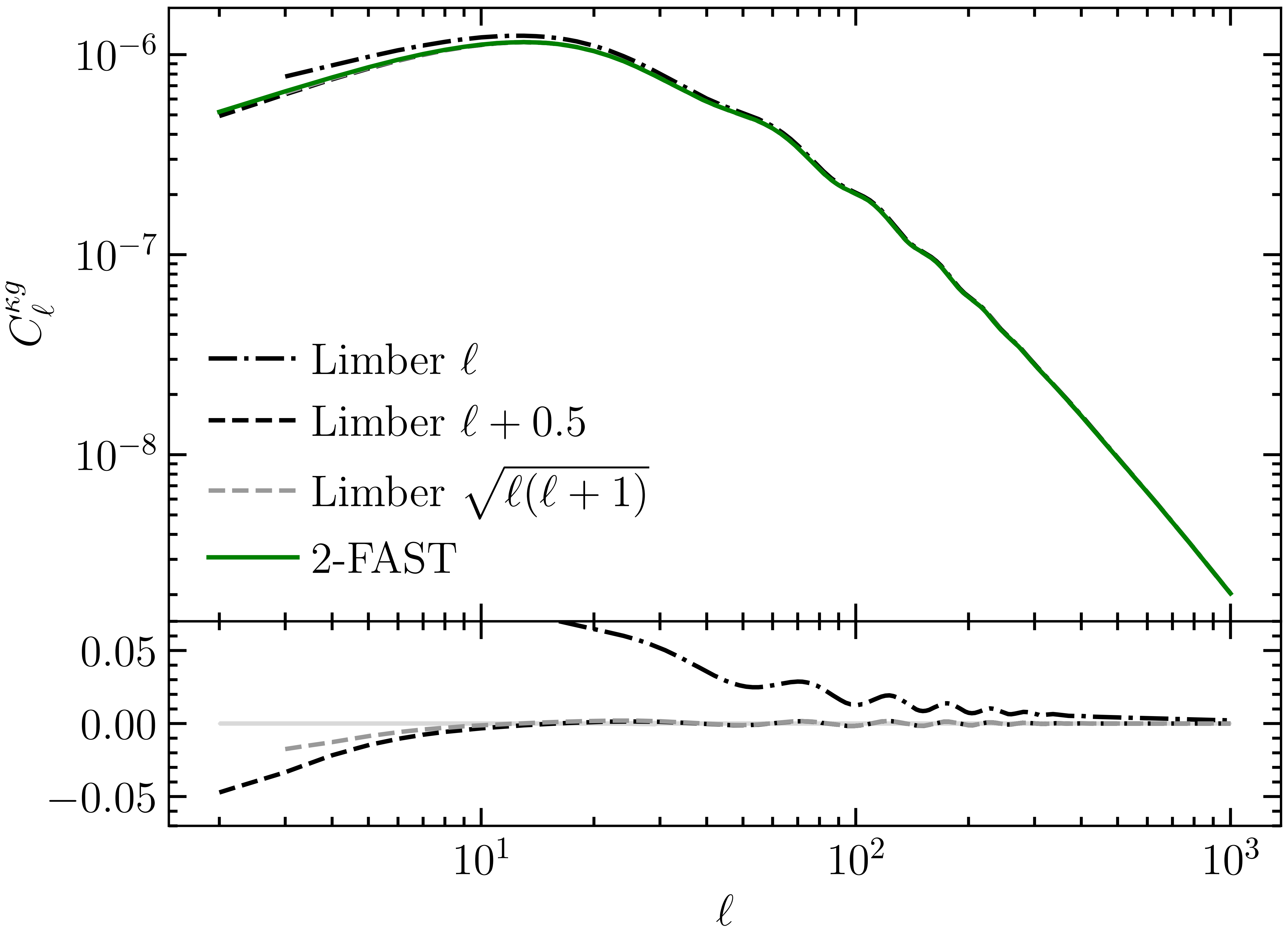
LoVerde,Afshordi 2008

$$C_\ell^{\psi\psi} = \int_0^1 \mathrm{d}R \int_0^{\chi_\star} \mathrm{d}\ln\chi \left[2\,\chi^2\,\varphi(\chi)\,\varphi(R\chi)\right] w_{\ell\ell}^p(\chi, R)\,. \quad (52)$$

Limber's approximation:

$$C_\ell^{\psi\psi} \approx \int_0^{\chi_\star} \mathrm{d}\chi \, \frac{\varphi^2(\chi)}{\chi^2} \, P\left(\frac{\nu}{\chi}\right)$$

Example: lensing-galaxy cross C_ℓ



$$C_\ell^{\kappa g}(\chi_\star, \chi') = \frac{3}{2}\Omega_m H_0^2\, \ell(\ell+1) \int_0^{\chi_\star/\chi'} d\ln R'\, \frac{\chi_\star - \chi}{\chi_\star} \times \frac{D(z)D(z')}{a} \times \left[b' w^p_{\ell,00}(\chi', R'\chi') - f' w^p_{\ell,20}(\chi', R'\chi')\right] \tag{60}$$

The lensing-galaxy cross correlation is the dominant contribution to the correlation between local galaxies and high-redshift galaxies.

Conclusion

Expressing projection integrals as convolution integrals allows the use of the Fast Fourier transform

Can precompute cosmology-independent parts

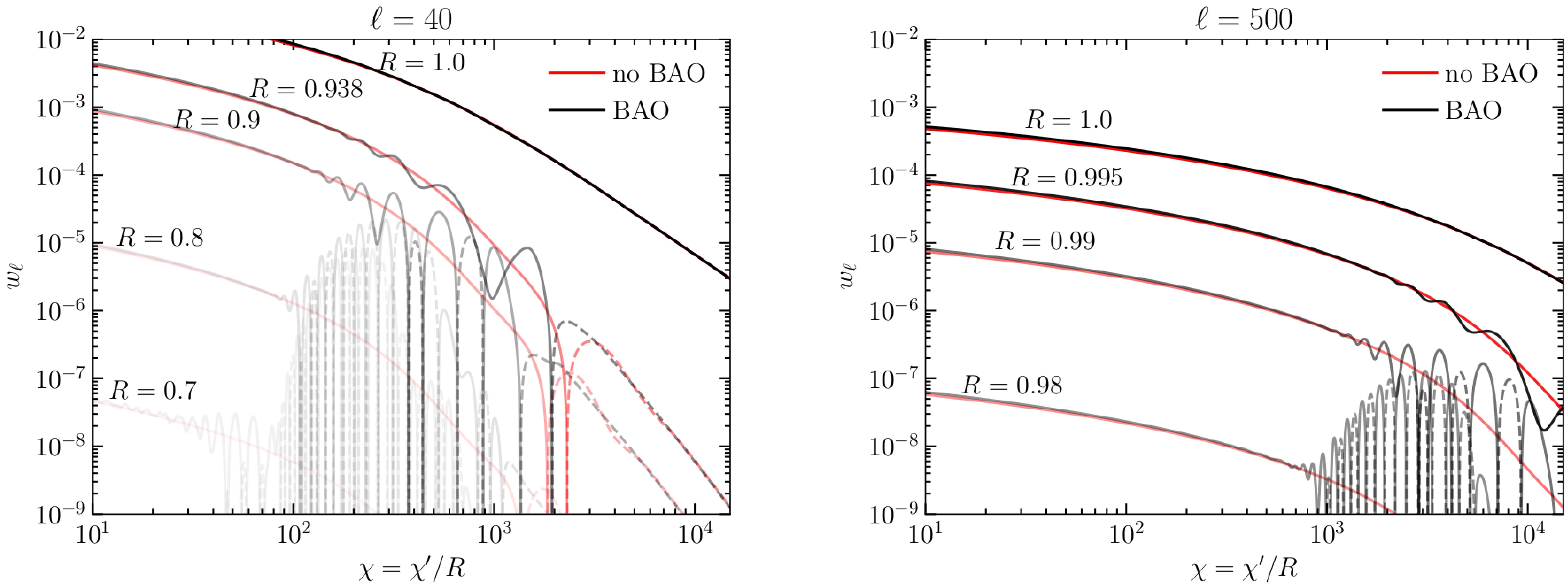
Useful for perturbation theory, and calculating auto- and cross-correlations

$$\xi_\ell^\nu(r) \equiv \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2} P(k) \frac{j_\ell(kr)}{(kr)^\nu}$$

$$w_{\ell\ell'}(\chi, \chi') = \frac{2}{\pi} \int_0^\infty \mathrm{d}k\, k^2\, P(k)\, j_\ell(k\chi)\, j_{\ell'}(k\chi')$$

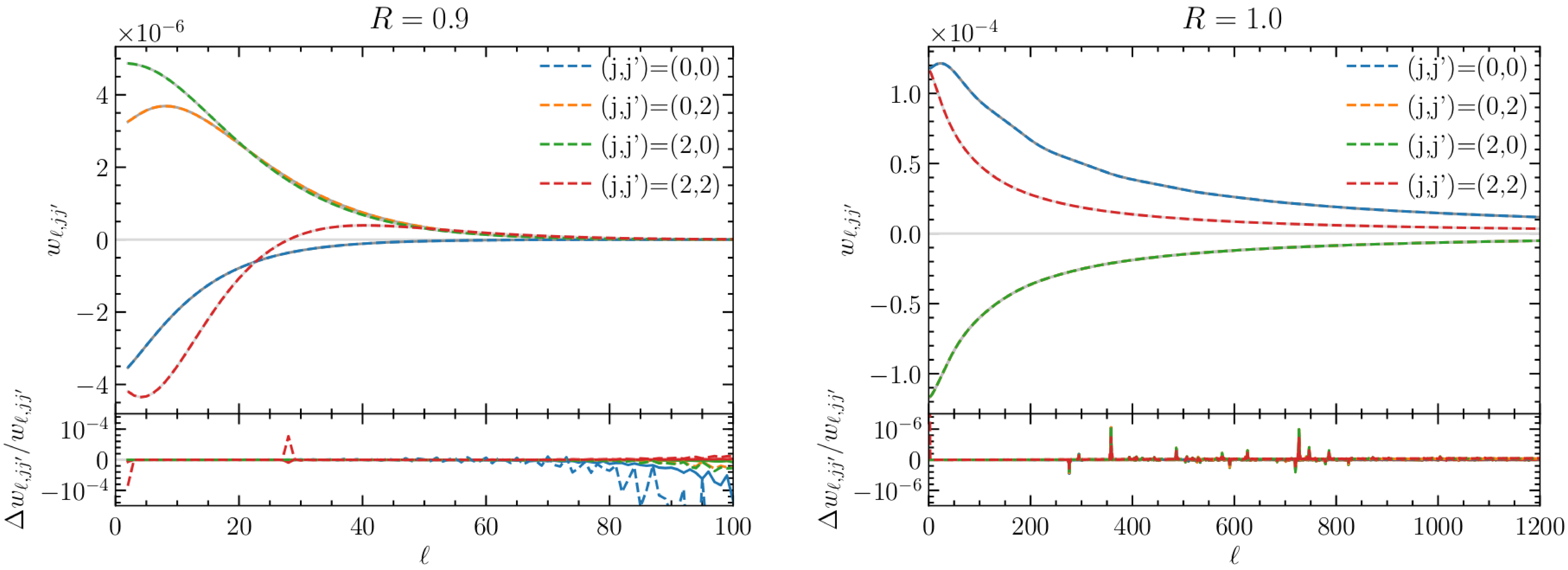
Backup slides

BAO or no BAO?



$\ell = 40$
$R = 1.0$
$R = 0.938$
$R = 0.9$
$R = 0.8$
$R = 0.7$
no BAO
BAO
w_ℓ
$\chi = \chi'/R$
$\ell = 500$
$R = 1.0$
$R = 0.995$
$R = 0.99$
$R = 0.98$
no BAO
BAO
w_ℓ
$\chi = \chi'/R$

w_ℓ as function of ℓ



We start from the integration of power spectrum overlappinig with one spherical Bessel function:

$$\xi_\ell^\nu(r) \equiv \int_0^\infty \frac{k^2 \mathrm{d}k}{2\pi^2} P(k) \frac{j_\ell(kr)}{(kr)^\nu} . \qquad (10)$$

We here briefly outline the method, and present some examples, including the calculation of the real-space correlation function $\xi(r) \equiv \xi_0^0(r)$ and its first and second derivatives.

The key observation is that, by introducing logarithmic variables κ and ρ such that

$$k = k_0 e^\kappa \qquad r = r_0 e^\rho , \qquad (11)$$

with some pivot k_0 and r_0, the integration in Eq. (10) can be expressed as a convolution:

$$\xi_\ell^\nu(r) = \frac{k_0^3 e^{-q\rho}}{2\pi^2 \alpha^\nu} \int_{-\infty}^\infty \mathrm{d}\kappa\, e^{(3-q)\kappa} P(k_0 e^\kappa) \times e^{(q-\nu)(\kappa+\rho)}\, j_\ell(\alpha e^{\kappa+\rho}) . \qquad (12)$$

Here, we define $\alpha = k_0 r_0$. That the convolution in real space is multiplication in Fourier space motivates us to introduce the Fourier transform of the spherical Bessel function $M_\ell^{\nu,q}(t)$:

$$e^{(q-\nu)\sigma}\, j_\ell(\alpha e^\sigma) = \int_{-\infty}^\infty \frac{\mathrm{d}t}{2\pi} e^{i\sigma t} M_\ell^{\nu,q}(t) , \qquad (13)$$

The first integral

with which and $\phi^q(t)$ that we defined earlier [Eq. (6)], Eq. (10) becomes

$$\xi_\ell^\nu(r) = \frac{k_0^3 e^{-q\rho}}{\pi \alpha^\nu} \int_{-\infty}^\infty \frac{\mathrm{d}t}{2\pi} e^{i\rho t}\, \phi^q(t)\, M_\ell^{\nu,q}(t) . \qquad (14)$$

$$\begin{aligned} M_\ell^{\nu,q}(t) &= \int_{-\infty}^\infty \mathrm{d}\sigma\, e^{-it\sigma}\, e^{(q-\nu)\sigma}\, j_\ell(\alpha e^\sigma) \\ &= \alpha^{it-q+\nu} \int_0^\infty \mathrm{d}s\, s^{q-\nu-1-it}\, j_\ell(s) \\ &\equiv \alpha^{it-q+\nu}\, u_\ell(q-\nu-1-it) . \end{aligned} \qquad (15)$$

The integral $u_\ell(n)$ is given by

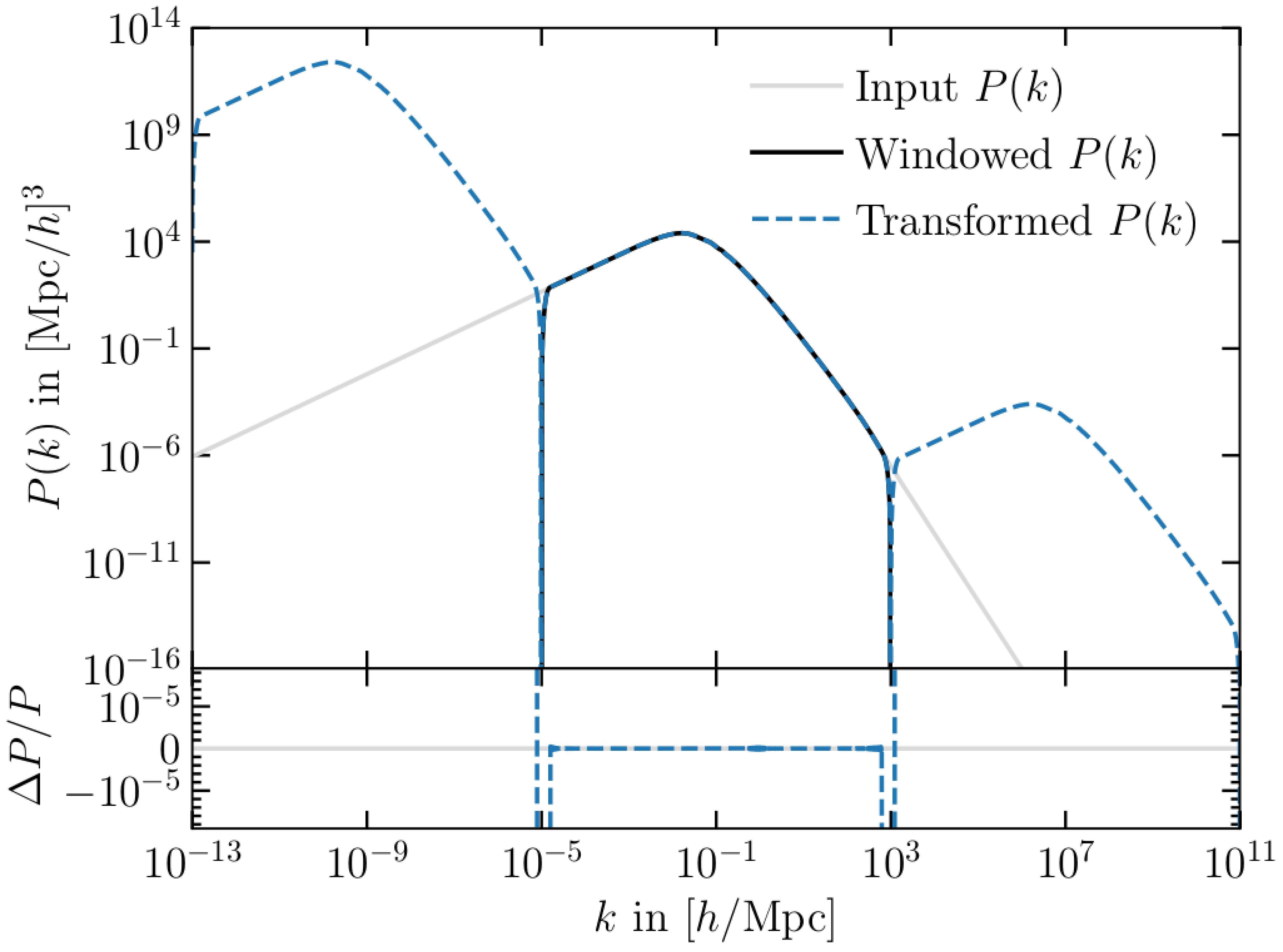
$$u_\ell(n) \equiv \int_0^\infty \mathrm{d}s\, s^n\, j_\ell(s) = 2^{n-1}\sqrt{\pi} \frac{\Gamma[(1+\ell+n)/2]}{\Gamma[(2+\ell-n)/2]} . \qquad (16)$$

Talman 1978

Key idea: Hankel transformation

$$\phi^q(x) = \int \frac{\mathrm{d}\kappa}{2\pi}\, e^{i\kappa x}\, e^{(3-q)\kappa}\, P(k_0 e^{\kappa})$$

$$P(k) = e^{-(3-q)\kappa} \int \mathrm{d}x\, e^{-i\kappa x}\, \phi^q(x)$$



- Operationally, it can be done by an FFT of the power spectrum sampled at regular intervals in log-*k* space. (FFTlog: Hamilton 2000)