Relation between field transformations and physical DOFs in scalar-tensor theories

Kazufumi Takahashi

RESCEU, The University of Tokyo
(JSPS fellow)

Based on

- **KT**, H. Motohashi, T. Suyama, and T. Kobayashi
  Phys. Rev. D **95**, 084053 (2017), "General invertible transformation and physical degrees of freedom"
- **KT** and T. Kobayashi
  arXiv:1708.02951, "Extended mimetic gravity: Hamiltonian analysis and gradient instabilities"

Scalar-tensor theories

- Scalar-tensor theories (inflation, late-time acceleration, …)
  - Higher derivatives ⋯ Ostrogradsky ghost

  "Any nondegenerate higher derivative theory contains extra ghost-like DOFs"

  e.g. $\det\left(\frac{\partial^2 L}{\partial \ddot{q}^i \partial \ddot{q}^j}\right) \neq 0$ for $L(q^i, \dot{q}^i, \ddot{q}^i)$ ➡ Need degeneracy!

- Scalar-tensor theories w/ 3 DOFs
  - Horndeski (generalized Galileons)
  - GLPV theories
  - quadratic/cubic DHOST (extended scalar-tensor) [*known broadest class*]

  Degenerate Higher-Order Scalar-Tensor

➡ Specify all the degenerate theories up to cubic order in $\nabla_\mu \nabla_\nu \phi$

$$S_{\mathrm{q/c}} = \int d^4x \sqrt{-g}\left[f_2 \mathcal{R} + f_3 \mathcal{G}^{\mu\nu} \nabla_\mu \nabla_\nu \phi + a^{\mu\nu\lambda\sigma} (\nabla_\mu \nabla_\nu \phi)(\nabla_\lambda \nabla_\sigma \phi) + b^{\mu\nu\lambda\sigma\alpha\beta} (\nabla_\mu \nabla_\nu \phi)(\nabla_\lambda \nabla_\sigma \phi)(\nabla_\alpha \nabla_\beta \phi)\right]$$

functions of (ϕ, X)

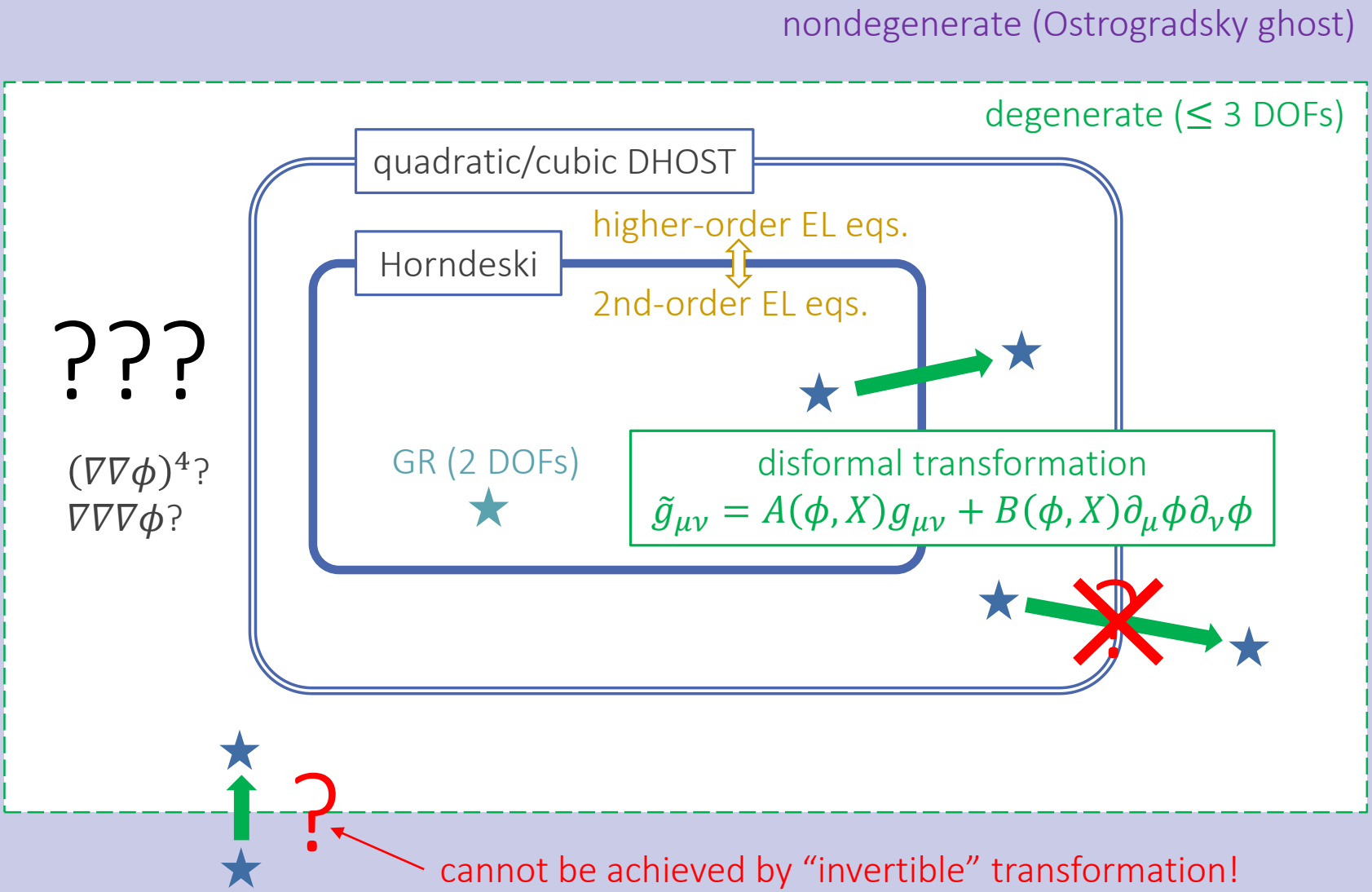
Chosen so that the Lagrangian is degenerate

$\mathcal{R}$: 4D Ricci scalar

$\mathcal{G}_{\mu\nu}$: 4D Einstein tensor

$X \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

➢ ST theories and disformal transformations



Invertible transformation

KT, Motohashi, Suyama, Kobayashi, 2017

- Write $\{g_{\mu\nu}, \phi\} \equiv \psi^i$ and consider $\tilde{\psi}^i = f^i(\psi^j, \partial_\mu\psi^j, \partial_\mu\partial_\nu\psi^j, \cdots)$

$$\delta\tilde{\psi}^i = \hat{P}^i_j \delta\psi^j, \qquad \hat{P}^i_j \equiv \frac{\partial f^i}{\partial\psi^j} + \frac{\partial f^i}{\partial(\partial_\mu\psi^j)}\partial_\mu + \frac{\partial f^i}{\partial(\partial_\mu\partial_\nu\psi^j)}\partial_\mu\partial_\nu + \cdots$$

derivative-operator matrix

- Relation between EL equations

$$\delta S = \int d^4x \tilde{\mathcal{E}}_i \delta\tilde{\psi}^i = \int d^4x \tilde{\mathcal{E}}_i (\hat{P}^i_j \delta\psi^j) = \int d^4x (\hat{P}^{\dagger i}{}_j \tilde{\mathcal{E}}_i)\delta\psi^j$$

$$\begin{cases} \tilde{\mathcal{E}}_i = 0\text{: original EL eqs.} \\ \mathcal{E}_i = 0\text{: new EL eqs.} \end{cases}$$

$$\mathcal{E}_j = \hat{P}^{\dagger i}{}_j \tilde{\mathcal{E}}_i$$

$$\boxed{\tilde{\mathcal{E}}_i = 0 \quad \Leftrightarrow \quad \mathcal{E}_i = 0}$$

same DOFs!

- Invertible: $\tilde{\psi}^i \leftrightarrow \psi^i \cdots$ unique inverse operator $\hat{Q}^i_j$:

$$\hat{P}^i_k \hat{Q}^k_j = \hat{Q}^i_k \hat{P}^k_j = \delta^i_j$$

$$\tilde{\mathcal{E}}_i = \hat{Q}^{\dagger j}{}_i \mathcal{E}_j$$

➡ No invertible trnsf. can map a nondegenerate ST theory to a degenerate one!
(4 DOFs) (3 DOFs)

➡ Noninvertible transformation!

➢ Way out

- How can we go outside the quadratic/cubic DHOST class?

  ➡ Let's start from nondegenerate theories!

invertible: $\tilde{g}_{\mu\nu} \leftrightarrow g_{\mu\nu}$ one-to-one

- If the transformation is invertible, the resultant theory is also nondegenerate

  (∵ DOFs are invariant under invertible trnsf.) KT, Motohashi, Suyama, Kobayashi, 2017

  ➡ Let's consider noninvertible transformations!

- Noninvertible conformal transformation

$$\tilde{g}_{\mu\nu} = -X g_{\mu\nu}$$

$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$X \to \Omega^{-2}X$ under $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$

$\Rightarrow$ $\tilde{g}_{\mu\nu}$: invariant

- Mimetic gravity

$$\tilde{S}[\tilde{g}_{\mu\nu}, \phi] \longrightarrow S[g_{\mu\nu}, \phi]$$

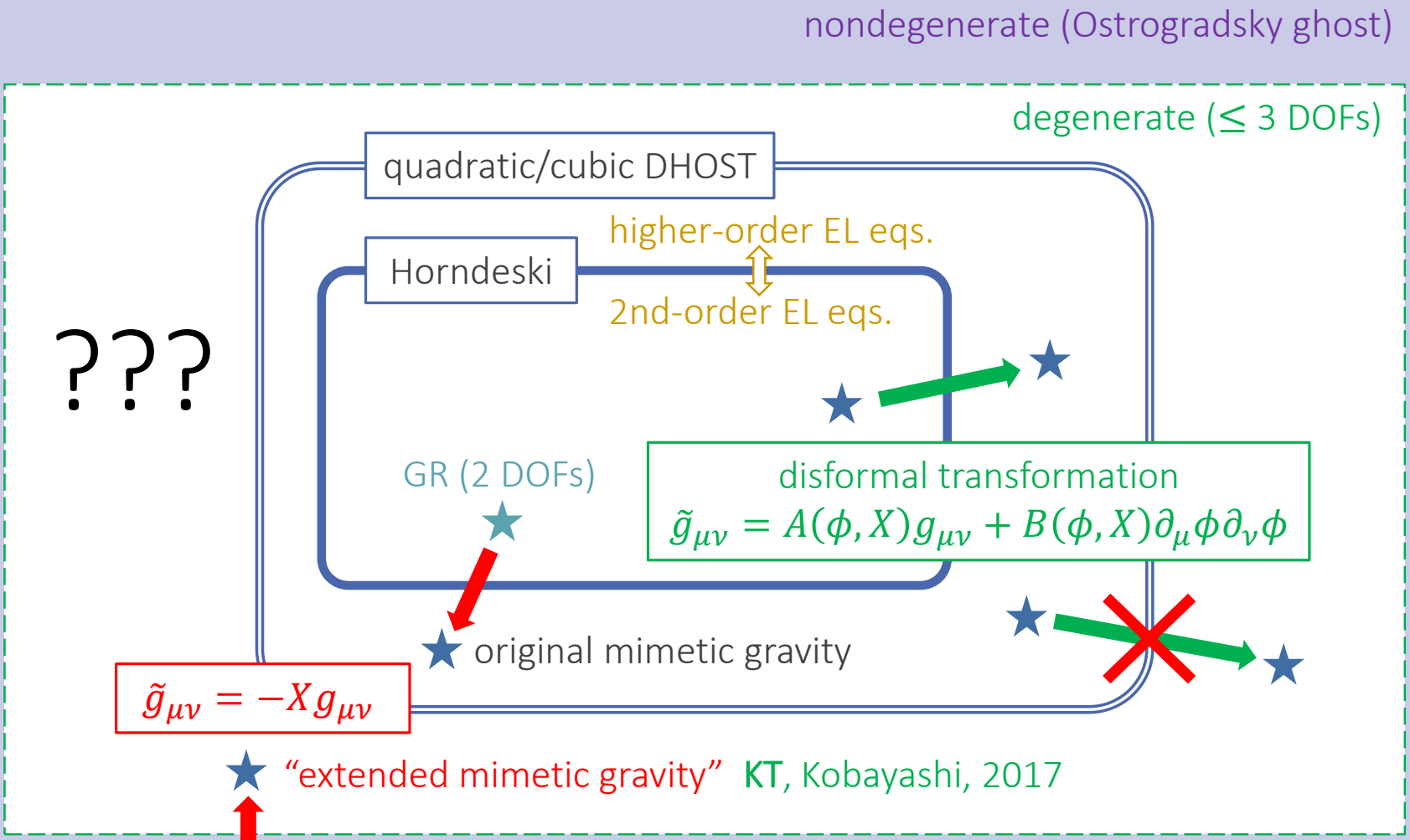
conformal sym.

"seed" ST theory → "mimetic theory"

- Originally, seed theory = GR

  proposed as a candidate of dark matter (Chamseddine, Mukhanov, 2013)

Mimetic theories



$$S = \int d^4x \sqrt{-g} \left[f_2 \mathcal{R} + f_3 \mathcal{G}^{\mu\nu} \nabla_\mu \nabla_\nu \phi + F\left(g_{\mu\nu}, \phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi\right) \right]$$

Seed theory

- Start from a "seed" action

$$S_{\text{seed}}[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[f_2\mathcal{R} + f_3\mathcal{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + F\left(g_{\mu\nu},\phi,\nabla_\mu\phi,\nabla_\mu\nabla_\nu\phi\right)\right]$$

- Known healthy theories amount to specific f_2, f_3, F:
  - GR: $f_2 = \frac{M_{\text{Pl}}^2}{2}, f_3 = 0, F = 0$
  - Horndeski: $\forall f_2, f_3$ functions of (ϕ, X)

$$F = -2f_{2X}\left[(\Box\phi)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2\right] + \frac{1}{3}f_{3X}\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right]$$

However, for generic choices of f_2, f_3, and F, the theory has 4 DOFs.

- Hamiltonian analysis ➡ 1+3 decomposition
  - First write the seed action in the ADM language and then move to the mimetic theory

1+3 decomposition

- ADM variables

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

- For ϕ, we define

canonical variable $$A_* \equiv n^\mu \nabla_\mu \phi = \frac{\dot{\phi} - N^i D_i \phi}{N}$$

velocity of A_* $$V_* \equiv n^\mu n^\nu \nabla_\mu \nabla_\nu \phi = \frac{\dot{A}_* - D^k \phi D_k N - N^k D_k A_*}{N}$$

$n_\mu \equiv -N\delta_\mu^0$
$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$
$K_{ij} \equiv \frac{1}{2N}\left(\dot{\gamma}_{ij} - 2D_{(i}N_{j)}\right)$
D_i: 3D covariant derivative

Derivatives of ϕ are decomposed as

$$\begin{cases} \nabla_\mu \phi = h_\mu^i D_i \phi - n_\mu A_* \\ \nabla_\mu \nabla_\nu \phi = h_{(\mu}^i h_{\nu)}^j \left(D_i D_j \phi - A_* K_{ij}\right) - 2h_{(\mu}^i n_{\nu)}\left(D_i A_* - K_{ij} D^j \phi\right) + n_\mu n_\nu V_* \end{cases}$$

→ $F\left(g_{\mu\nu}, \phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi\right)$ contains $\gamma_{ij}, \phi, A_*, K_{ij}, V_*$, and D_i

1+3 decomposition (cont'd)

- Terms with the curvature tensors:

$$\int d^4x\sqrt{-g}\left[f_2\mathcal{R} + f_3\mathcal{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi\right]$$
$$= \int dtd^3xN\sqrt{\gamma}\Big\{f_2\left(R + K_{ij}^2 - K^2\right) - 2Kf_{2\perp} - 2D_iD^if_2 - \frac{1}{2}\left(R - K_{ij}^2 + K^2\right)A_*f_{3\perp}$$
$$-\left[R_{ij} - \frac{1}{2}\left(R + K_{kl}^2 - K^2\right)\gamma_{ij}\right]D^i\phi D^jf_3 + D_iD_j\left(D^i\phi D^jf_3\right) - D_iD^i\left(D_j\phi D^jf_3\right)$$
$$+\left(K\gamma^{ij} - K^{ij}\right)\left(2K_i^kD_k\phi D_jf_3 + f_{3\perp}D_iD_j\phi + A_*D_iD_jf_3\right) + \Lambda\left(NA_* + N^iD_i\phi - \dot{\phi}\right)\Big\}$$

Fixes $A_* = \frac{\dot{\phi} - N^iD_i\phi}{N}$

with

$$f_\perp \equiv n^\mu\nabla_\mu f = f_\phi A_* - 2f_X\left(K_{ij}D^i\phi D^j\phi + A_*V_* - D^i\phi D_iA_*\right)$$

- Combined with the term $F\left(g_{\mu\nu}, \phi, \nabla_\mu\phi, \nabla_\mu\nabla_\nu\phi\right)$, we obtain

$$\tilde{S}_{\text{seed}}[\tilde{g}_{\mu\nu}, \phi] = \int dtd^3x\left[\tilde{N}\sqrt{\tilde{\gamma}}L_0\left(\tilde{\gamma}_{ij}, \tilde{R}_{ij}, \phi, \tilde{A}_*; \tilde{K}_{ij}, \tilde{V}_*; \tilde{D}_i\right) + \Lambda\left(\tilde{N}\tilde{A}_* + \tilde{N}^i\tilde{D}_i\phi - \dot{\phi}\right)\right]$$

→ Perform $\tilde{g}_{\mu\nu} = -Xg_{\mu\nu}$ $\qquad X = -A_*^2 + D_i\phi D^i\phi$

Extended mimetic gravity

- Under $\tilde{g}_{\mu\nu} = -X g_{\mu\nu}$,

$$\tilde{N} = \sqrt{-X}N, \qquad \tilde{N}^i = N^i, \qquad \tilde{\gamma}_{ij} = -X\gamma_{ij}, \qquad \tilde{A}_* = \frac{1}{\sqrt{-X}}A_*,$$

$$\tilde{R}_{ij} = R_{ij} + \frac{3}{4X^2}D_iXD_jX - \frac{1}{2X}D_iD_jX + \gamma_{ij}\left(\frac{1}{4X^2}D_kXD^kX - \frac{1}{2X}D_kD^kX\right),$$

while the velocities are transformed as

$$\tilde{K}_{ij} = \sqrt{-X}\left[\left(\delta_i^k\delta_j^l - \frac{D^k\phi D^l\phi}{X}\gamma_{ij}\right)V_{kl} + \frac{D^k\phi D_kA_*}{X}\gamma_{ij}\right],$$

$$\tilde{V}_* = -\frac{1}{X^2}D^i\phi D^j\phi\left(A_*V_{ij} - D_iD_j\phi\right),$$

Only in the combination of

$$V_{ij} \equiv K_{ij} + \frac{V_*}{A_*}\gamma_{ij}$$

$$\Rightarrow S_{\mathrm{mimetic}}[g_{\mu\nu},\phi] = \int dt d^3x\left[N\sqrt{\gamma}L_{\mathrm{M}}\left(\gamma_{ij}, R_{ij}, \phi, A_*; V_{ij}; D_i\right) + \Lambda\left(NA_* + N^iD_i\phi - \dot{\phi}\right)\right]$$

ADM representation of “extended mimetic gravity”

Hamiltonian analysis

$$S[g_{\mu\nu},\phi] = \int dt d^3x \left[N\sqrt{\gamma} L_{\mathrm{M}}\left(\gamma_{ij}, R_{ij}, \phi, A_*; B_{ij}; D_i\right) + \Lambda\left(NA_* + N^i D_i \phi - \dot{\phi}\right) + N\lambda^{ij}\left(B_{ij} - V_{ij}\right)\right]$$

auxiliary field

$$B_{ij} = V_{ij}$$

- Canonical variables ⋯ 50-dim. phase space

$$\begin{pmatrix} N & N^i & \gamma_{ij} & \phi & A_* & B_{ij} & \Lambda & \lambda^{ij} \\ \pi_N & \pi_i & \pi^{ij} & p_\phi & p_* & p^{ij} & P & P_{ij} \end{pmatrix}$$

- Primary constraints

$$\pi_N \approx 0, \qquad \pi_i \approx 0, \qquad p^{ij} \approx 0, \qquad P \approx 0, \qquad P_{ij} \approx 0$$

$$\bar{\pi}^{ij} \equiv \pi^{ij} + \frac{1}{2}\lambda^{ij} \approx 0, \qquad \bar{p}_\phi \equiv p_\phi + \Lambda \approx 0$$

and

$$\mathcal{C} \equiv A_* p_* - 2\gamma_{ij}\pi^{ij} \approx 0$$

generator of conformal transformation

$$V_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} + 2\frac{\dot{A}_*}{A_*}\gamma_{ij}\right) + \cdots$$

- Redefine $\mathcal{C}$ for convenience:

$$\bar{\mathcal{C}} \equiv \mathcal{C} + 2\lambda^{ij} P_{ij}$$

➢ Hamiltonian analysis (cont'd)

- Total Hamiltonian

Primary constraints

$$H_T = \int d^3x \left(N\mathcal{H} + N^i\mathcal{H}_i + \mu_N\pi_N + \mu^i\pi_i + \mu_{ij}\bar{\pi}^{ij} + u_\phi\bar{p}_\phi + u_*\bar{\mathcal{C}} + u_{ij}p^{ij} + UP + U^{ij}P_{ij} \right)$$

with

$$\mathcal{H} \equiv -\sqrt{\gamma}L_{\mathrm{M}}(\gamma_{ij}, R_{ij}, \phi, A_*; B_{ij}; D_i) + 2\pi^{ij}B_{ij} + p_\phi A_* - \sqrt{\gamma}D_i\left(\frac{p_*}{\sqrt{\gamma}}D^i\phi\right)$$

$$\mathcal{H}_i \equiv -2\sqrt{\gamma}D^j\left(\frac{\pi_{ij}}{\sqrt{\gamma}}\right) + p_\phi D_i\phi + p_* D_i A_* + p^{jk}D_i B_{jk} - 2\sqrt{\gamma}D_j\left(\frac{p^{jk}}{\sqrt{\gamma}}B_{ik}\right)$$

generate spatial diffeo. of $\gamma_{ij}, \phi, A_*, B_{ij}$

- Secondary constraints

$$\dot{\pi}_N \approx 0 \quad \rightarrow \quad \mathcal{H} \approx 0$$

$$\dot{\pi}_i \approx 0 \quad \rightarrow \quad \mathcal{H}_i \approx 0$$

$$\dot{p}^{ij} \approx 0 \quad \rightarrow \quad \varphi^{ij} \equiv \sqrt{\gamma}\frac{\partial L_{\mathrm{M}}}{\partial B_{ij}} - 2\pi^{ij} \approx 0$$

- No tertiary constraint if $\det\left(\frac{\partial^2 L_{\mathrm{M}}}{\partial B_{ij}\partial B_{kl}}\right) \neq 0$

DOF counting

- First-class constraints ⋯ 9 in total

$$\underbrace{\pi_N, \quad \pi_i}_{\text{4D diffeo.}}, \quad \bar{\mathcal{C}}, \quad \underbrace{\mathcal{H}, \quad \mathcal{H}_i}_{\text{EM conservation}}$$

conformal

- Second-class constraints ⋯ 26 in total

$$\bar{\pi}^{ij}, \quad \bar{p}_\phi, \quad p^{ij}, \quad P, \quad P_{ij}, \quad \varphi^{ij}$$

- The number of DOFs

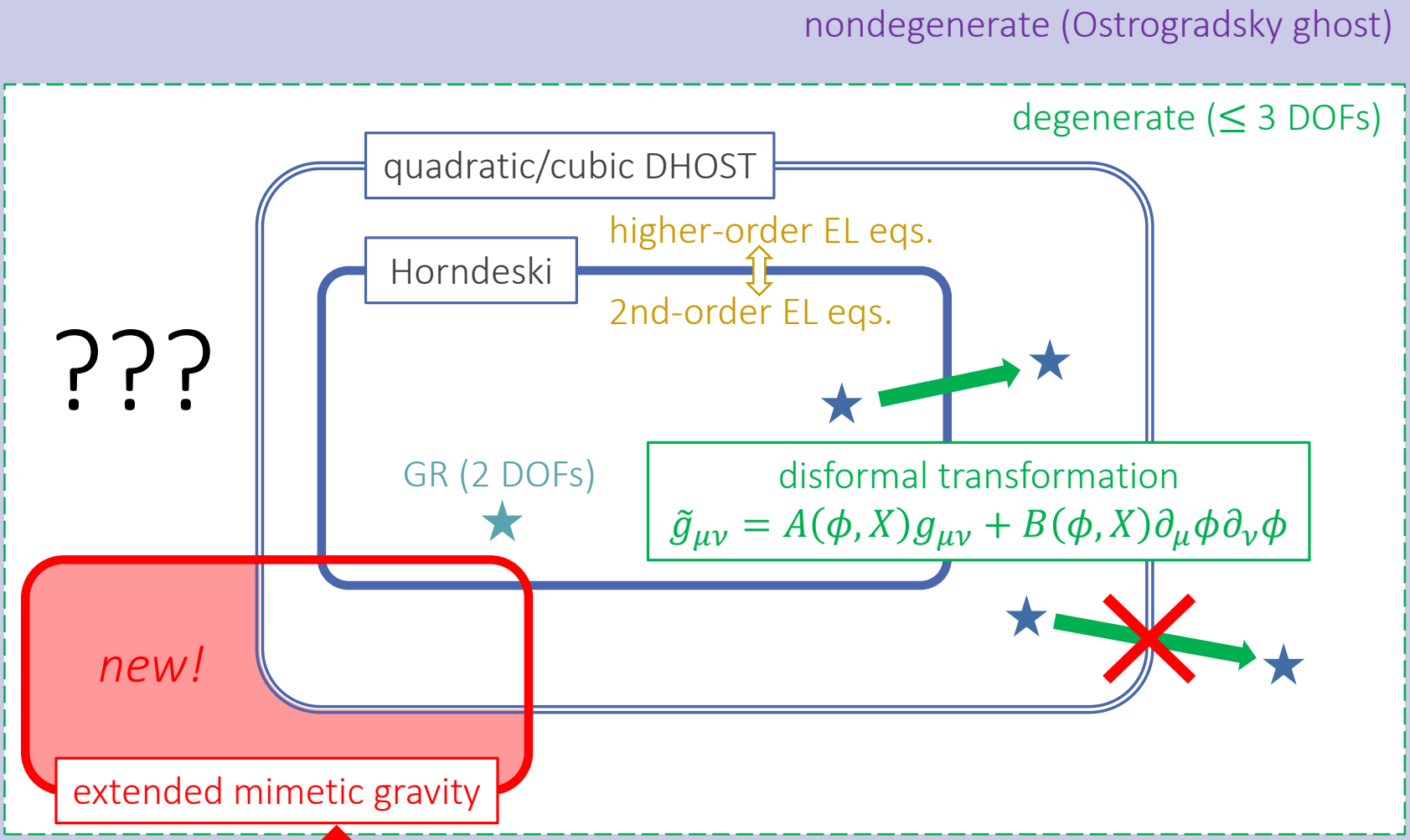
$$\frac{1}{2}(50 - 9 \times 2 - 26) = 3$$

No extra DOF!

phase-space dim. # of first-class # of second-class

- If there are tertiary constraints, the number of DOFs can become even smaller.

Relation to the known classes



$$\tilde{g}_{\mu\nu} = -X g_{\mu\nu} \qquad S = \int d^4x \sqrt{-\tilde{g}} \left[f_2 \tilde{\mathcal{R}} + f_3 \tilde{\mathcal{G}}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + F\left(\tilde{g}_{\mu\nu}, \phi, \tilde{\nabla}_\mu \phi, \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \right]$$

➢ Cosmological perturbations

- $\phi = t$ and $X = -1$ (fix the conformal gauge DOF)

[See also Shin'ichi Hirano's talk]

$$S = \int dt d^3x \sqrt{\gamma}\left[\left(f_2 - \frac{1}{2}\dot{f}_3\right)R + \mathcal{F}(t, K, \mathcal{K}_2, \mathcal{K}_3, \cdots, \mathcal{K}_\ell)\right], \qquad \mathcal{K}_n \equiv K^{i_1}_{i_2}K^{i_2}_{i_3}\cdots K^{i_n}_{i_1}$$

- Metric ansatz

$$N = 1, \qquad N_i = \partial_i \chi, \qquad \gamma_{ij} = a^2(t)e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{kj}\right)$$

scalar pert. TT tensor pert.

- Background EOM

$$\frac{d\mathcal{P}}{dt} + 3H\mathcal{P} - \mathcal{F} = 0, \qquad \mathcal{P} \equiv \sum_{n=1}^{\ell} nH^{n-1}\mathcal{F}_n, \qquad \mathcal{F}_n \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{K}_n}$$

where the arguments of $\mathcal{F}$ and $\mathcal{F}_n$ are evaluated on the background: $\mathcal{K}_n = 3H^n$

- Tensor quadratic action

$$S_{\mathrm{T}}^{(2)} = \int dt d^3x \frac{a^3}{4}\left[\mathcal{B}\dot{h}_{ij}^2 - \mathcal{E}\frac{(\partial_k h_{ij})^2}{a^2}\right]$$

Stable if $\mathcal{B} > 0, \mathcal{E} > 0$

with

$$\mathcal{B} \equiv \sum_{n=2}^{\ell} \frac{n(n-1)}{2} H^{n-2}\mathcal{F}_n, \qquad \mathcal{E} \equiv f_2 - \frac{1}{2}\dot{f}_3$$

➢ Cosmological perturbations (cont'd)

- Scalar quadratic action [See also Shin'ichi Hirano's talk]

$$S_{\mathrm{S}}^{(2)} = \int dt d^3x a^3 \left[\frac{3}{2}(3\mathcal{A}+2\mathcal{B})\dot{\zeta}^2 - 2\mathcal{E}\frac{(\partial_k\zeta)^2}{a^2} + \frac{1}{2}(\mathcal{A}+2\mathcal{B})\left(\frac{\partial^2\chi}{a^2}\right)^2 - (3\mathcal{A}+2\mathcal{B})\dot{\zeta}\frac{\partial^2\chi}{a^2}\right]$$

with

$$\mathcal{A} \equiv \sum_{m=1}^{\ell}\sum_{n=1}^{\ell} mnH^{m+n-2}\mathcal{F}_{mn} \qquad \mathcal{F}_{mn} \equiv \frac{\partial^2\mathcal{F}}{\partial\mathcal{K}_m\partial\mathcal{K}_n}$$

χ can be eliminated by use of its EOM:

$$\frac{\partial^2\chi}{a^2} = \frac{3\mathcal{A}+2\mathcal{B}}{\mathcal{A}+2\mathcal{B}}\dot{\zeta}$$

Assume $\mathcal{A}+2\mathcal{B} \neq 0$

$$\Rightarrow S_{\mathrm{S}}^{(2)} = \int dt d^3x a^3 \left[\frac{2(3\mathcal{A}+2\mathcal{B})}{\mathcal{A}+2\mathcal{B}}\mathcal{B}\dot{\zeta}^2 + 2\mathcal{E}\frac{(\partial_k\zeta)^2}{a^2}\right]$$

Stable if

$$\frac{2(3\mathcal{A}+2\mathcal{B})}{\mathcal{A}+2\mathcal{B}}\mathcal{B} > 0, \qquad \mathcal{E} < 0$$

Conflict with $\mathcal{E} > 0$ for tensor modes!

- If $\mathcal{B}(\mathcal{A}+2\mathcal{B})(3\mathcal{A}+2\mathcal{B}) = 0$, scalar perturbations become nondynamical

➢ Conclusions

- How to go beyond quadratic/cubic DHOST theories via disformal transformation?

  ➡ Consider noninvertible transformation of nondegenerate theories!

  (∵ Invertible transformations cannot change the DOFs) KT, Motohashi, Suyama, Kobayashi, 2017

- "Extended mimetic gravity" KT, Kobayashi, 2017

Perform $\tilde{g}_{\mu\nu} = -X g_{\mu\nu}$ on theories with 4 DOFs:

$$\tilde{S}[\tilde{g}_{\mu\nu}, \phi] = \int d^4x \sqrt{-\tilde{g}} \left[f_2 \tilde{\mathcal{R}} + f_3 \tilde{\mathcal{G}}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + F\left(\tilde{g}_{\mu\nu}, \phi, \tilde{\nabla}_\mu \phi, \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \right]$$

functions of $(\phi, \tilde{X})$ $\quad \tilde{X} \equiv \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

➡ The resultant theory $S[g_{\mu\nu}, \phi]$ has only 3 DOFs due to conformal sym.

… However, it does not admit viable cosmology

Appendices

Invertible disformal transformation

- Disformal transformation ⋯ maps a ST theory to another ST theory

$$\tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi$$

original metric new metric

- If A and B satisfy

$$A(A + XB)(A - XA_X - X^2 B_X) \neq 0,$$

then $\tilde{g}_{\mu\nu} \leftrightarrow g_{\mu\nu}$: one-to-one, i.e.,

$$g_{\mu\nu} = \tilde{A}(\phi, \tilde{X})\tilde{g}_{\mu\nu} + \tilde{B}(\phi, \tilde{X})\partial_\mu\phi\partial_\nu\phi,$$

$$\tilde{A}(\phi, \tilde{X}) = \frac{1}{A(\phi, X)}, \qquad \tilde{B}(\phi, \tilde{X}) = -\frac{B(\phi, X)}{A(\phi, X)}, \qquad \tilde{X} = \frac{X}{A + XB}$$

- For example,

$$\tilde{g}_{\mu\nu} = X^2 g_{\mu\nu} + X\partial_\mu\phi\partial_\nu\phi,$$

is uniquely solved for $g_{\mu\nu}$ as

$$g_{\mu\nu} = 4\tilde{X}^2\tilde{g}_{\mu\nu} - 2\tilde{X}\partial_\mu\phi\partial_\nu\phi.$$

In general, the solution cannot be obtained in a closed form…

➢ Noninvertible disformal transformation

- Any generic noninvertible disformal transformation

$$\tilde{g}_{\mu\nu} = A(\phi, X) g_{\mu\nu} - \left(\frac{A(\phi, X)}{X} + f(\phi) \right) \partial_\mu \phi \partial_\nu \phi, \qquad f: \text{arbitrary}$$

can be reduced to a noninvertible conformal transformation if composed with

$$g_{\mu\nu} = \left[\bar{Q} + \frac{\bar{X} B(\phi, \bar{X}/\bar{Q})}{A(\phi, \bar{X}/\bar{Q})} \right] \bar{g}_{\mu\nu} - \frac{B(\phi, \bar{X}/\bar{Q})}{A(\phi, \bar{X}/\bar{Q})} \partial_\mu \phi \partial_\nu \phi, \qquad \bar{Q}(\phi, \bar{X}) \neq q(\phi) \bar{X}$$

Indeed, $\tilde{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are related as

$$\tilde{g}_{\mu\nu} = -f(\phi) \bar{X} \bar{g}_{\mu\nu},$$

where $f(\phi)$ can be absorbed into the redefinition of ϕ.

Mimetic constraint

- We defined the mimetic theories by noninvertible conformal transformation

  ➡ Original-frame scalar kinetic term suffers from $\tilde{X} = -1$

- Alternative formulation of mimetic gravity ⋯ impose the "mimetic constraint"

$$S_{\text{mimetic}} = S[g_{\mu\nu}, \phi] + \int d^4x \sqrt{-g}\lambda(X+1)$$

Fixes $X = -1$

- From the Lagrangian viewpoint, one can show that the two formulations give equivalent sets of EOMs.

- From the Hamiltonian viewpoint, the additional mimetic constraint can be regarded as a gauge condition that completely fixes the local conformal gauge DOF of the mimetic gravity.

➢ Unitary gauge

- Choose t so that $\phi = t \cdots \partial_\mu \phi$ should be timelike

  ➡ automatically satisfied by the mimetic constraint:

$$X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1 < 0$$

- $\phi = t$ & $X = -1$ imply $N = 1$. Therefore, the action of the mimetic gravity is simplified as

$$S_{\text{unitary}} = \int dt d^3x \sqrt{\gamma} L(\gamma_{ij}, R_{ij}, t; K_{ij}; D_i)$$

  - Canonical variables $(N^i, \gamma_{ij}; \pi_i, \pi^{ij}) \cdots$ 18-dim. phase space
  - Primary constraints $\pi_i \approx 0$ lead to the momentum constraints $\mathcal{H}_i \approx 0$. They are 1st-class.

➡ $\frac{1}{2}(18 - 6 \times 2) = 3$ DOFs

Consistent with the result
obtained w/o unitary gauge fixing!