Kinetic theory for fermions in curved space-time

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Kinetic theory for photons in one slide

- A gas of particles is described statistically
- Distribution function is central object $f(\eta, x^i, p^i)$
- Dynamics is given by Boltzmann equation

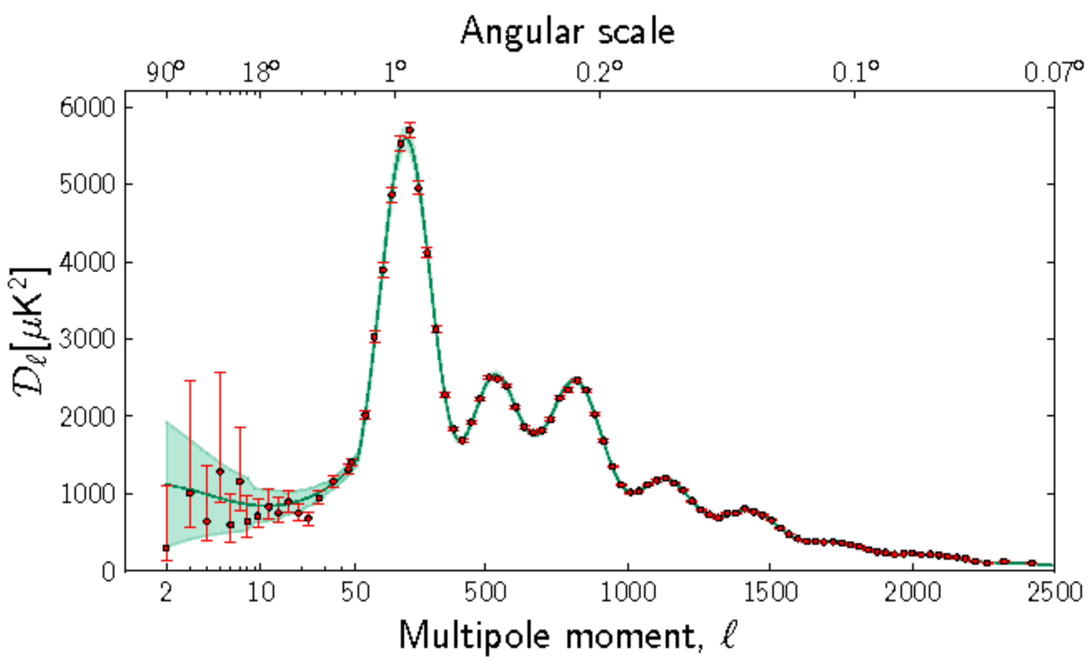
$$\underbrace{L[f]}_{\text{Free streaming}} = \underbrace{C[f]}_{\text{Collisions}}$$

- Polarization is described by tensor valued distribution function $f_{\mu\nu}$

$$\boxed{\frac{1}{2}\begin{pmatrix} I+Q & U-\mathrm{i}V \\ U+\mathrm{i}V & I-Q \end{pmatrix}}$$

Main results for CMB

- Boltzmann transport combined with fluid description for baryons and GR
- Observation of both intensity and polarization multipoles
- Silk damping at small scales is due partly to polarization effects



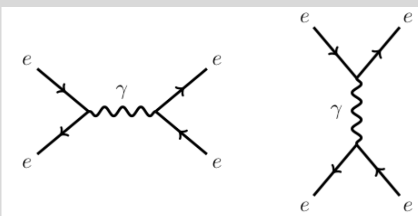
Motivations

- How does the kinetic description works for Fermions ?
- What is polarization and how is it described ?
- What is the structure of free streaming and collisions ?
- When does it matter ?

Relativity and QFT

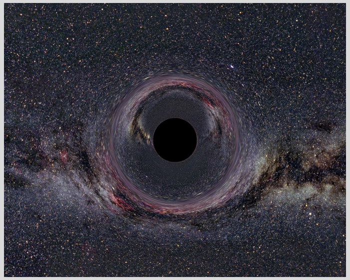
- Collisions with electrons : Quantum Mechanics
- Transport along non-trivial geodesics : General Relativity

Microscopic Scale



Mesoscopic Scale

Macroscopic Scale



Well established for photons and used for CMB !

Statistics from the quantum world

- Number operator from annihilation/creation operators of helicity states

$$N_{rs}(p,p') \equiv a_r^\dagger(p) a_s(p')$$

- Distribution function associated to a quantum state

$$\langle\Psi|N_{rs}(p,p')|\Psi\rangle \equiv \underline{\delta}(p-p')\boxed{f_{rs}(p)}$$

Quantum Transport Equations

$$\underline{\delta}(0)\frac{\mathrm{d}}{\mathrm{d}t}f_{rs} = \langle\Psi|\frac{\mathrm{d}N_{rs}}{\mathrm{d}t}|\Psi\rangle = i\langle\Psi|[H,N_{rs}]|\Psi\rangle \equiv \underline{\delta}(0)\,C[f_{rs}(t,p)]$$

Computation of $C[f_{ss'}(t,p)]$ involves typical scattering matrix elements weighted by the distribution functions

Construction of covariant distribution function

- Case of Bosons: use of polarization basis ϵ_r^μ

$$f^{\mu\nu}(p) \equiv \sum_{r,s=-1,0,1} f_{rs}(p)\epsilon_r^{\star\mu}(\boldsymbol{p})\epsilon_s^\nu(\boldsymbol{p})$$

Spacetime indices

Polarization basis for photons is transverse, spatially projected

-> Stokes parameters

- Case of Fermions : plane wave solutions of Dirac equations $u_s\,,\,v_s$

Spinor indices

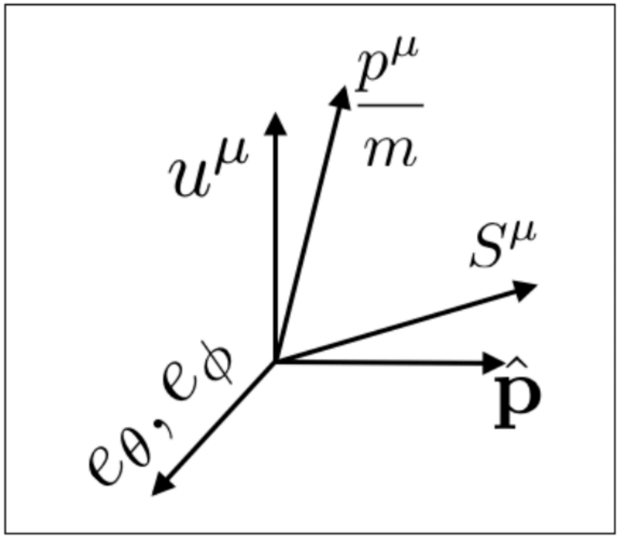
$$\boxed{F_{\mathfrak{a}}{}^{\mathfrak{b}}(p) \equiv \begin{cases} \sum_{rs} f_{rs}(p)u_{s,\mathfrak{a}}(p)\bar{u}_r^{\mathfrak{b}}(p) & \text{particles,} \\ \sum_{rs} f_{rs}(p)v_{r,\mathfrak{a}}(p)\bar{v}_s^{\mathfrak{b}}(p) & \text{antiparticles.} \end{cases}}$$

How to get rid of spinor indices ???

Let us use $\boxed{\{\mathbb{1}, \gamma^\mu, \Sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}}$

$$\Sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$$

Covariantization is simple



- We define the parameters
  - Intensity $I \equiv f_{++} + f_{--}$
  - Circular polarisation $V \equiv f_{++} - f_{--}$
  - Linear polarisation $Q_\pm \equiv \sqrt{2} f_{\pm\mp}$
  - Polarisation vector $Q^\mu \equiv Q_+ \epsilon_+^\mu + Q_- \epsilon_-^\mu$
  - The combined vector $\mathcal{Q}^\mu \equiv Q^\mu + V S^\mu$
- F can be decomposed as $\boxed{F = \underbrace{\frac{I}{2}}_{\text{1 dof}}(M + \not{p}) + \frac{M}{2}\gamma^5 \gamma_\mu \underbrace{\mathcal{Q}^\mu}_{\text{3 dof}} + p^\mu \tilde{\Sigma}_{\mu\nu} \mathcal{Q}^\nu}$

  with $\widetilde{\Sigma}^{\mu\nu} \equiv -\frac{\gamma^5}{4}[\gamma^\mu, \gamma^\nu]$

- I and $\mathcal{Q}^\mu$ are individually observer independent , while both helicity and linear polarisation are not.

Structure of decomposition $N_{rs}(p,p') \equiv a_r^\dagger(p) a_s(p')$

- Boson of spin 1 : SO(3) $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$
- Massless boson : SO(2) $\mathbf{2}_1 \otimes \mathbf{2}_1 = \underbrace{\mathbf{2}_2}_{\text{Polarization}} \oplus \underbrace{\mathbf{1}}_{V} \oplus \underbrace{\mathbf{1}}_{I}$

- Fermions spin ½ : SO(3) $\mathbf{2} \otimes \mathbf{2} = \underbrace{\mathbf{3}}_{Q^\mu} \oplus \underbrace{\mathbf{1}}_{I}$

Liouville equation for intensity

$$\frac{\partial I(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial\tau}+\frac{P^i}{P^0}\frac{\partial I(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial x^i}+\frac{\mathrm{d}p^i}{\mathrm{d}\tau}\frac{\partial I(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial p^i}=0$$

Relativistic Boltzmann Equation

$$\begin{aligned}\frac{\partial Q^\alpha(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial\tau} &+ \frac{P^i}{P^0}\frac{\partial Q^\alpha(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial x^i}+\frac{\mathrm{d}p^i}{\mathrm{d}\tau}\frac{\partial Q^\alpha(\tau,\boldsymbol{x},\boldsymbol{p})}{\partial p^i}\\ &+ \Gamma^\alpha_{\gamma\beta}\frac{P^\gamma}{P^0}Q^\beta(\tau,\boldsymbol{x},\boldsymbol{p})=\frac{E}{P^0}C^\alpha_Q\end{aligned}$$

Spherical Harmonics decomposition

I (and V) are scalars

$$I(\boldsymbol{p}) = \sum_{\ell m} I_{\ell m}(|\boldsymbol{p}|) Y_{\ell m}(\hat{\mathbf{p}}) \,.$$

Linear Polarization is spin-1

$$Q_+(\boldsymbol{p}) \equiv \sum_{\ell m} Q^+_{\ell m}(|\boldsymbol{p}|) Y^+_{\ell m}(\hat{\mathbf{p}}) \,, \quad Q_-(\boldsymbol{p}) \equiv Q^-_{\ell m}(|\boldsymbol{p}|) Y^-_{\ell m}(\hat{\mathbf{p}}) \,.$$

Fermi theory of weak interactions

- Current-current interactions (Fermions only)

$$\mathcal{H}_I = -\mathcal{L}_I = g\left(J^{ac}_\mu J^\mu_{bd} + \mathrm{cc}\right)$$

- Structure of reactions is $a + b \leftrightarrow c + d$

Collision term generates (linear) polarization

Transverse projector

$$Q^i_{C[A]}(p) \propto m_a \int \frac{[\mathrm{d}p_c][\mathrm{d}p_d]}{E_b} \delta(E_c + E_d - E - E_b)(\delta^i_j + \hat{p}^i \hat{p}_j)(p^j_c + p^j_d) I_c(\boldsymbol{p}_c) I_d(\boldsymbol{p}_d)(p_c \cdot p_d)$$

- Suppression by m/E
- Requires dipolar structure in distribution function

Flavor oscillations

[31] G. Sigl and G. Raffelt, Nuclear Physics **B406**, 423 (1993).

Non-diagonal mass matrix $$\Omega_{ij} = \sqrt{\boldsymbol{p}^2 + M^2}_{ij}$$

$$\frac{\mathrm{d}f_{ij}(t,p)}{\mathrm{d}t} = \boxed{\mathrm{i}\sum_k \left(\Omega_{ik}(p) f_{kj}(t,p) - f_{ik}(t,p)\Omega_{kj}(p)\right)} + C[f_{ij}(t,p)] \,.$$

Thanks