Theoretical Consistency of Stochastic Approach

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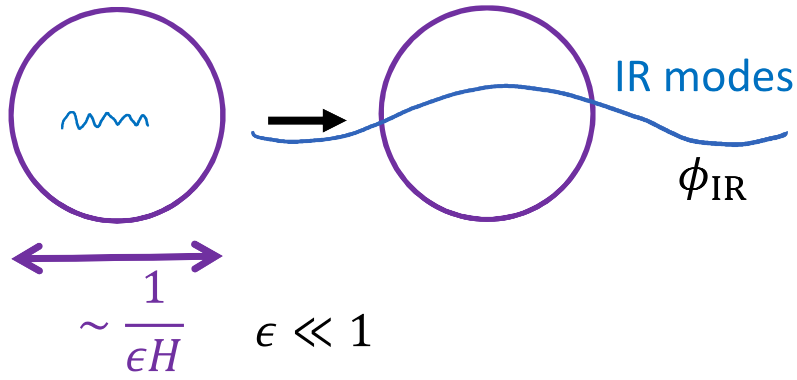
Observables = QFT expectation values ??

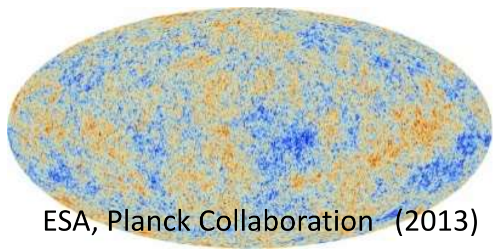
Quantum fluctuations of an inflaton ϕ

$\downarrow$

long–wavelength (IR) modes
well outside the horizon.



Cosmic Microwave Background (CMB)


ESA, Planck Collaboration (2013)

Correlation function $< \phi_{\mathrm{IR}} \phi_{\mathrm{IR}} >$

***Observables* = <u>*QFT expectation values* $< \phi_{IR} \phi_{IR} >$</u>**

?

IR divergences

loop corrections→IR divergences/ secular growth

ϕ: a minimally coupled massless scalar in de Sitter space

QFT expectation values suffer from...

- IR divergences
- cutoff-dependent secular growth

  (introducing an artificial cutoff $k = a_0 H$)

e.g. $\lambda\phi^4$ theory $(\lambda \ll 1)$ $(a(t) \propto e^{Ht})$

$<\phi_{\mathrm{IR}}^2(x)>$ $k \geq a_o H$: an IR cutoff

$\ln\frac{a}{a_0} \sim t - t_0$

$$
\begin{aligned}
&\sim \ln\frac{a}{a_0} + 1 \\
&+\lambda\left[\left(\ln\frac{a}{a_0}\right)^3 + \left(\ln\frac{a}{a_0}\right)^2 + \left(\ln\frac{a}{a_0}\right) + 1\right] \\
&+\lambda^2\left[\left(\ln\frac{a}{a_0}\right)^5 + \left(\ln\frac{a}{a_0}\right)^4 + \left(\ln\frac{a}{a_0}\right)^3 + \cdots\right] \\
&+\cdots
\end{aligned}
$$

leading order(LO) Sub-LO

We need **a consistent prescription of calculating observables,** which is free from IR divergences.

One candidate:

Stochastic Approach

Stochastic Formalism recovers LO secular growth

Stochastic Formalism

A. A. Starobinsky (1986) A. A. Starobinsky and J. Yokoyama(1994)
V. Vennin and A. A. Starobinsky(2015)

$$\dot{\phi}_{\mathrm{IR}} = \underbrace{-\frac{1}{3H}V'(\phi_{\mathrm{IR}})}_{\text{deterministic}} + \underbrace{\xi}_{\text{stochastic}} \qquad <\xi(x_1)\xi(x_2)> = \frac{H^3}{4\pi^2}\delta(t_1-t_2)\frac{\sin(\epsilon a(t_1)H|\vec{x_1}-\vec{x_2}|)}{\epsilon a(t_1)H|\vec{x_1}-\vec{x_2}|}$$

ϕ_{IR}: coarse-grained field ϵaH: coarse-graining scale

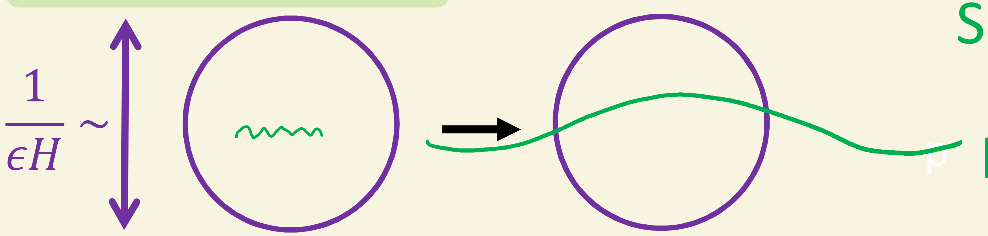
Brownian motion with an external force

✓ LO secular growth terms are correctly recovered.

N. C. Tsamis and R. P. Woodard (2005)

✓ This eq. can be solved non-perturbatively.

Physical origin of ξ



Short-wavelength (UV) modes

are transferred to

IR modes (phase is random)

Stochastic force

Stochastic Picture of inflationary universe

Stochastic Approach A.Linde(1986) A. A. Starobinsky (1986)

The time evolution of inflationary universe = **classical stochastic process**

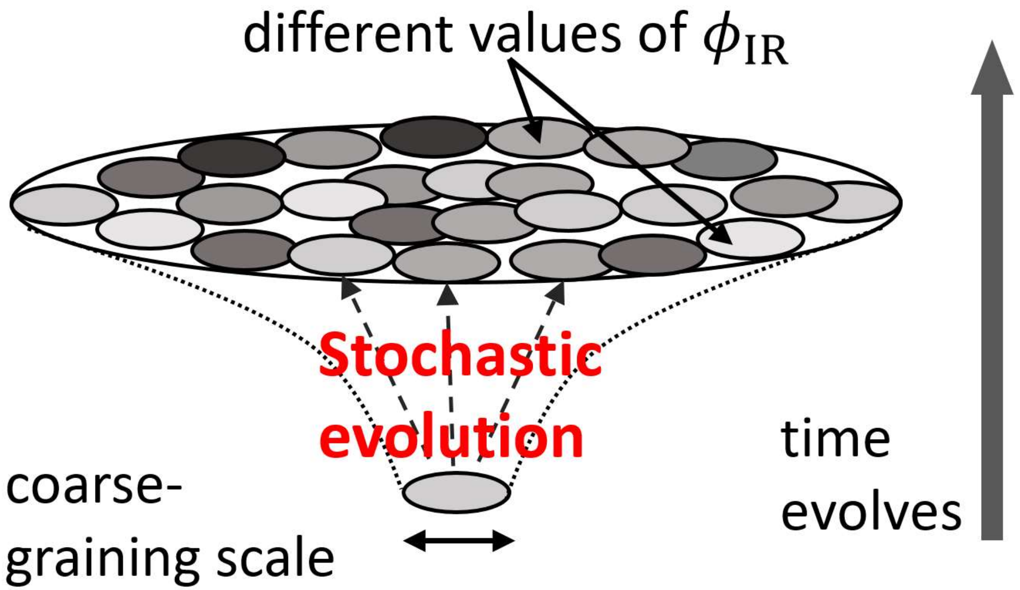
Brownian Motion

In this approach,

LO secular growth

||

an increase of the statistical variance.



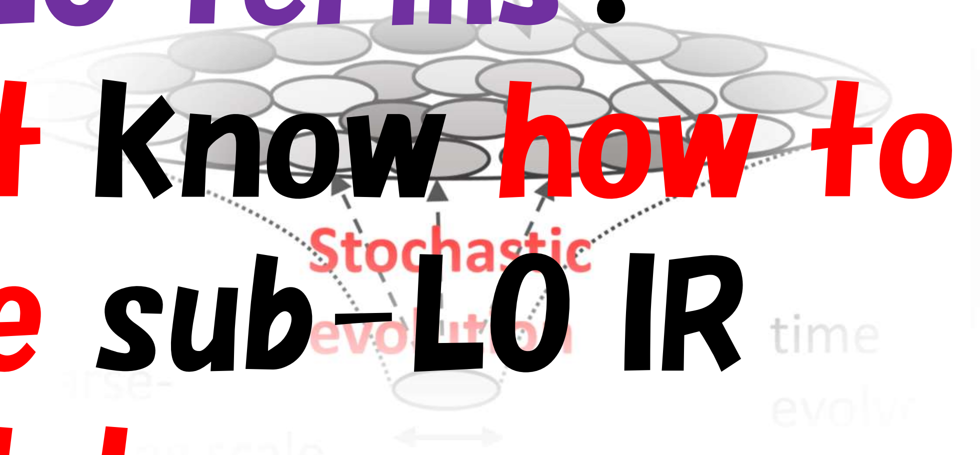
✓ The prescription of calculating adiabatic perturbations based on this picture, which is free from IR div is proposed.

T. Fujita *et al.*(2013) V. Vennin and A. A. Starobinsky(2015)

Stochastic Picture of inflationary universe

Stochastic Approach A.Linde(1986) A. A. Starobinsky (1986)

- **Does this classical stochastic picture hold for sub-LO terms?**
- **We do not know how to regularize sub-LO IR divergent terms.**



✓ The prescription of calculating adiabatic perturbations based on this picture, which is free from IR div. is proposed.

T. Fujita *et al.*(2013) V. Vennin and A. A. Starobinsky(2015)

The nature of sub-LO secular growth is unclear

$\lambda\phi^4$ theory ($\lambda \ll 1$)

$$< \phi_{\mathrm{IR}}^2(x) >$$

$$\begin{aligned} &\sim \ln\frac{a}{a_0} + 1 \\ &+\lambda\left[\left(\ln\frac{a}{a_0}\right)^3 + \left(\ln\frac{a}{a_0}\right)^2 + \left(\ln\frac{a}{a_0}\right) + 1\right] \\ &+\lambda^2\left[\left(\ln\frac{a}{a_0}\right)^5 + \left(\ln\frac{a}{a_0}\right)^4 + \left(\ln\frac{a}{a_0}\right)^3 + \cdots\right] \\ &+\cdots \end{aligned}$$

LO
(Leading order)

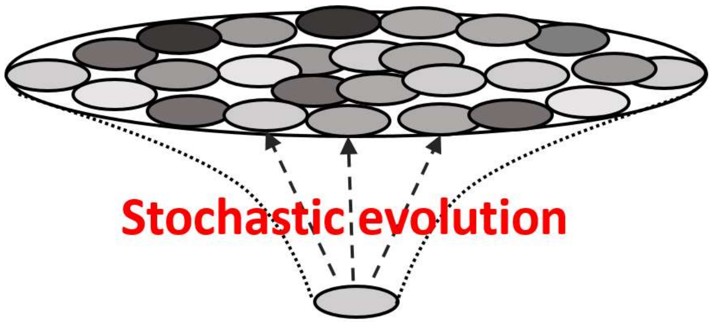
Sub-LO

What kind of equation describes ***all*** secular growth terms?

- Classical stochastic process?
- Long-time (spatial) correlations?

Qualitative difference?

compared to Brownian motion



If this stochastic picture breaks down,

the current picture of inflationary universe might be drastically modified.

Our Work

$\lambda\phi^4$ theory ($\lambda \ll 1$)

$$< \phi_{\mathrm{IR}}^2(x) >$$

$$\begin{aligned}
&\sim \ln\frac{a}{a_0} + 1 \\
&+\lambda\left[\left(\ln\frac{a}{a_0}\right)^3 + \left(\ln\frac{a}{a_0}\right)^2 + \left(\ln\frac{a}{a_0}\right) + 1\right] \\
&+\lambda^2\left[\left(\ln\frac{a}{a_0}\right)^5 + \left(\ln\frac{a}{a_0}\right)^4 + \left(\ln\frac{a}{a_0}\right)^3 + \cdots\right] \\
&+\cdots
\end{aligned}$$

LO NLO Sub-LO

(Next-to-leading order)

As a first step, we study the theory with a massless scalar on *de Sitter background*

1. Formulate a systematic way of deriving an effective EoM for IR modes which can describe all IR secular growth terms.

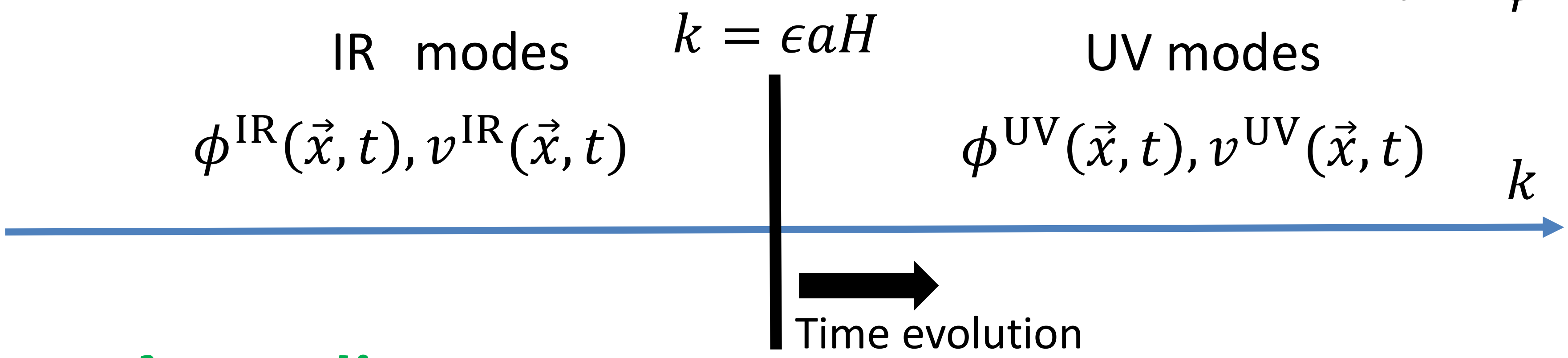
2. Derive an effective EoM which correctly recovers secular growth to NLO.

can be seen as a classical stochastic process.

Setup

- **Model** : $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi)$ on de Sitter background
- **Defs. of UV modes and IR modes**

$v \equiv \dot{\phi}$



- **Assumptions**
  1. No IR mode initially (at $t = t_0$)
  2. V(ϕ) is turned on at $t = t_0$, and take the Bunch-Davies vacuum states for a free field at $t = t_0$
- **Strategy**

Derive an effective EoM for IR modes **by integrating out UV modes**

⇨ Path integral is useful

M. Morikawa (1990)

Decompose the path integral: bilinear interaction

We want to
integrate out UV modes

Decompose the path integral

$$\int \mathcal{D}\phi\mathcal{D}v\, \mathrm{e}^{iS_0} = \int \mathcal{D}\phi^{\mathrm{IR}}\mathcal{D}v^{\mathrm{IR}}\mathrm{e}^{iS_0^{\mathrm{IR}}} \int \mathcal{D}\phi^{\mathrm{UV}}\mathcal{D}v^{\mathrm{UV}}\mathrm{e}^{iS_0^{\mathrm{UV}}} \boxed{\mathrm{e}^{iS_{\mathrm{bilinear}}}}$$

Interaction term which describes UV → IR transition

(IR propagator) (interaction vertex) (UV propagator)

By introducing
the bilinear interaction term which describes UV → IR transition,

we can decompose the path integral into UV parts and IR parts.

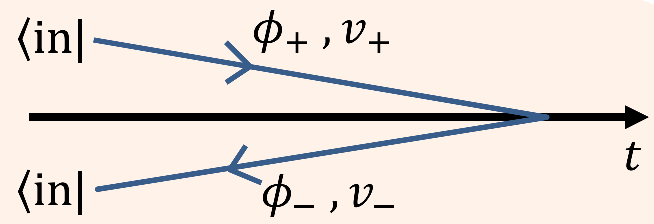
Derive an effective EoM for IR modes

◆Generating functional for IR modes $Z[J^{\mathrm{IR}}]$

$$Z[J^{\mathrm{IR}}]$$

$$= \int \mathcal{D}\phi_c^{\mathrm{IR}} \mathcal{D}\phi_\Delta^{\mathrm{IR}} \mathcal{D}v_c^{\mathrm{IR}} \mathcal{D}v_\Delta^{\mathrm{IR}} \mathrm{e}^{iS_0^{\mathrm{IR}} + iJ_\phi^{\mathrm{IR}}\cdot\phi^{\mathrm{IR}} + iJ_v^{\mathrm{IR}}\cdot v^{\mathrm{IR}}} \boxed{\int \mathcal{D}\phi_c^{\mathrm{UV}} \mathcal{D}\phi_\Delta^{\mathrm{UV}} \mathcal{D}v_c^{\mathrm{UV}} \mathcal{D}v_\Delta^{\mathrm{UV}} \mathrm{e}^{iS_0^{\mathrm{UV}}} \boxed{\mathrm{e}^{iS_{\mathrm{bilinear}}}} \mathrm{e}^{iS_{\mathrm{self}}}} \equiv e^{i\Gamma'}$$

$$\phi_c \equiv \frac{\phi_+ + \phi_-}{2}, \quad \phi_\Delta \equiv \phi_+ - \phi_-$$



◆ An effective EoM for a given $e^{i\Gamma'}$

$e^{i\Gamma'}$ determines
the nature of these terms

$$e^{i\Gamma'} \longrightarrow e^{i[\phi_\Delta^{\mathrm{IR}}(\cdots)+v_\Delta^{\mathrm{IR}}(\cdots)]}$$

$f(x) = \int dk\, \tilde{f}(k) e^{ikx}$
linear in x

$$\int \mathcal{D}\phi_\Delta^{\mathrm{IR}} \mathcal{D}v_\Delta^{\mathrm{IR}}$$

$$\dot{\phi}_c^{\mathrm{IR}} = v_c^{\mathrm{IR}} + \mu(\phi_c^{\mathrm{IR}}) + \xi_\phi$$

$$\dot{v}_c^{\mathrm{IR}} = -3Hv_c^{\mathrm{IR}} - V'_{\mathrm{eff}}(\phi_c^{\mathrm{IR}}) + \xi_v$$

effective EoM

NLO＝*Field-dep. random walk*

- Up to NLO, IR secular growth can be described by

$$\boxed{\dot{\phi}_c^{\rm IR} = v_c^{\rm IR} + \xi_\phi, \quad \dot{v}_c^{\rm IR} = -3Hv_c^{\rm IR} - \frac{\lambda {\phi_c^{\rm IR}}^3}{6}}$$

$$\left\langle \xi_\phi(x_1)\xi_\phi(x_2)\right\rangle = \underbrace{\boxed{\frac{H}{4\pi^2}\left[\underbrace{H^2}_{\text{LO}} + \underbrace{\lambda {\phi_c^{\rm IR}}^2(x_1)\left(l(\epsilon)-\frac{1}{9}\right)}_{\text{NLO}}\right]}}_{\equiv N}\delta(t_1-t_2)j_0(\epsilon a(t_1)H|\vec{x_1}-\vec{x_2}|)$$

$$l(\epsilon) := -\frac{1}{3}\left[\ln\frac{1}{\epsilon} - \ln 2 - \gamma + 2\right]$$

Difference from the LO case

Amplitude of the noise depends on ϕ_c^{IR}

Time / Spatial correlation : the same as the LO case

- The weight function P of the stochastic noise is positive definite at least up to NLO (as long as N is positive.)

$$P[\xi_\phi;\phi_c^{\rm IR}] = \frac{1}{\sqrt{2\pi N}}e^{-\frac{\xi_\phi^2}{2N}} > 0 \qquad <\xi_\phi\cdots\xi_\phi> = \int d\xi_\phi P[\xi_\phi;\phi_c^{\rm IR}]\xi_\phi\cdots\xi_\phi$$

Summary

◆Motivation

- IR loops of light scalars ・・・ IR divergences→ Need to regularize
- LO secular growth・・・an increase of statistical variance

→stochastic interpretation regularizes LO divergences.

Sub-LO can be regularized in the same manner?
can be described by a classical stochastic process?

◆Our Work

1. Formulate a systematic way of deriving an effective EoM for IR modes which can describe all IR secular growth terms.
2. Derive an effective EoM which correctly recovers secular growth to NLO, which can be seen as a classical stochastic process.